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The effective resistance

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Contents

Introduction

In this thesis we take a closer look at the effective resistance on a graph. The motivation for this is that we live in a world where there are networks found at any given place or time. For these networks it is very important to keep performing well when they are subject to attacks or failures. Here the effective resistance gives more understanding about the performance of these networks, also it gives more understanding in other fields of mathematics such as both algebraic and differential topology. I have written a small section dedicated to applications in these fields of mathematics (see section 3).

In the first section we provide ourselves with some useful tools to investigate different formulas for the effective resistance. We will do this by looking at two important subspaces of a vector space determined by the edges of the graph [3]. We will go through some well known results and give a full proof of the very famous Matrix Tree Theorem. With these mathematical results we will further investigate both known and lesser known formulas for the effective resistance, and also show the connections between some of them. One of these connections provides geometrical arguments by looking at the projection on the two subspaces from section 1. Finally we also take a more probabilistic approach in order to discover more about the effective resistance.

1 Linear algebra on graphs

In this section a graph G is an ordered pair (V, E) of vertices and edges, with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$ a subset of $V \times V \backslash \Delta_V$ where $\Delta_V = \{(v, v) | v \in V\}$ denotes the diagonal of $V \times V$. So we look at finite directed graphs without loops or multiple edges. Also we consider G to be connected, i.e., there is a path (see DEFINITION 2.4) between every couple of vertices. Furthermore we will write uv as a shorthand for $(u, v) \in E(G)$.

Now that we stated our graph we want to define a couple of vector spaces on the graph. First we denote by $\mathbb{R}^V$ the *vertex space* of all $\mathbb{R}-$valued functions from the vertex set $V(G)$ of the graph G into $\mathbb{R}$. Similarly we denote by $\mathbb{R}^E$ the *edge space* of $\mathbb{R}-$valued functions from the edge set $E(G)$ of G into $\mathbb{R}$. If we look at the function $\delta_{v_i} \in \mathbb{R}^V$ which is 0 everywhere, except at v_i, where it is 1, then $\delta_{v_1}, \delta_{v_2}, \ldots, \delta_{v_n}$ forms a basis for $\mathbb{R}^V$ and elements $\mathbf{x} = \sum_{i=1}^{n} x_i \delta_{v_i}$ are usually written in the form $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Now it is easy to see that $\mathbb{R}^V$ and $\mathbb{R}^E$ are indeed vector spaces with dimensions equal to $\#V = n$ respectively $\#E = m$. Furthermore we shall endow these spaces with the dot product.

First we want to take a closer look at the edge space $\mathbb{R}^E$. We will start to define two subspaces which will later turn out to be orthogonal subspaces.

Definition. 1.1. A *cycle* L of G is a graph such that $V(L) = \{u_1, u_2, \ldots, u_k\} \subset V(G)$ and $E(L) = \{u_1u_2, u_2u_3, \ldots, u_ku_1\}$ such that for every $uv \in E(L)$ we have $uv \in E(G)$ or $vu \in E(G)$.

Every cycle L of G can be identified with an element $\mathbf{z}_L \in \mathbb{R}^E$:

$$\text{For every } uv \in E(G) \text{ we have } \mathbf{z}_L(uv) = \begin{cases} 1 \text{ if } uv \in E(L), \\ -1 \text{ if } vu \in E(L), \\ 0 \text{ otherwise.} \end{cases}$$

Now define the *cycle space* $Z(G)$ to be the subspace of $\mathbb{R}^E$ which is spanned by the elements $\mathbf{z}_L$ as L runs over the set of cycles.

Definition. 1.2. A *cut* of G is the set $(V_1, V_2) = \{uv \in E(G) | u \in V_1, v \in V_2\}$ where $\{V_1, V_2\}$ is a partition of $V(G)$, i.e., $V_1 \cup V_2 = G$, $V_1 \cap V_2 = \emptyset$.

Every cut associated with a partition P of G can be identified with an element $\mathbf{s}_P \in \mathbb{R}^E$:

$$\text{For every } uv \in E(G) \text{ we have } \mathbf{s}_P(uv) = \begin{cases} 1 \text{ if } uv \in (V_1, V_2), \\ -1 \text{ if } vu \in (V_1, V_2), \\ 0 \text{ otherwise.} \end{cases}$$

Now define the *cut space* $S(G)$ to be the subspace of $\mathbb{R}^E$ which is spanned by the elements $\mathbf{s}_P$ as P runs over the set of partitions.

Theorem 1. *The inner product space $\mathbb{R}^E$ is equal to the orthogonal direct sum of $Z(G)$ and $S(G)$. Also we have $\text{Dim}(Z(G)) = m-n+1$ and $\text{Dim}(S(G)) = n-1$.*

Proof. First we want to check whether $Z(G)$ and $S(G)$ are orthogonal. Let $\mathbf{z}_L \in Z(G)$ and $\mathbf{s}_P \in S(G)$ where L is a cycle of G and $P = (V_1, V_2)$ a partition, then we have

$$\langle \mathbf{z}_L, \mathbf{s}_P \rangle = \#\{uv \in L|\ uv \in (V_1, V_2)\} - \#\{uv \in L|\ uv \in (V_2, V_1)\} = 0.$$

Thus we have that $Z(G) \perp S(G)$. Now since we know that $\text{Dim}(\mathbb{R}^E) = \#E(G) = m$ it is sufficient to show that $\text{Dim}(Z(G)) \geq m-n+1$ and $\text{Dim}(S(G)) \geq n-1$. Let T be a spanning tree of G and choose the indices of the edges in T such that $e_1, e_2, \ldots, e_{n-1}$ are the tree edges and $e_n, e_{n+1}, \ldots, e_m$ are the chords of T. We know for every tree edge e_i that if we would erase it from T the remainder of the spanning tree falls into two components. Let V_1^i be the vertex set of the component containing the initial vertex of e_i and let $V_2^i = V \backslash V_1^i$. If we now let $P_i = V_1^i \cup V_2^i$ be the corresponding partition then it is easy to see that for $1 \leq j \leq n-1$ we have $\mathbf{s}_{P_i}(e_j) = \delta_{ij}$. The cut (V_1^i, V_2^i) is the *fundamental cut* belonging to e_i and T, and $\mathbf{s}_{P_i}$ is called the *fundamental cut vector* belonging to e_i and T. For every chord e_i we also know that if we would add it to our tree T we would get a canonical cycle C_i of G that is oriented in the same direction as e_i. For $n \leq j \leq m$ it holds that $\mathbf{z}_{C_i}(e_j) = \delta_{ij}$. Here we call the cycle C_i the *fundamental cycle* belonging to e_i and T, and $\mathbf{z}_{C_i}$ the *fundamental cycle vector* belonging to e_i and T.

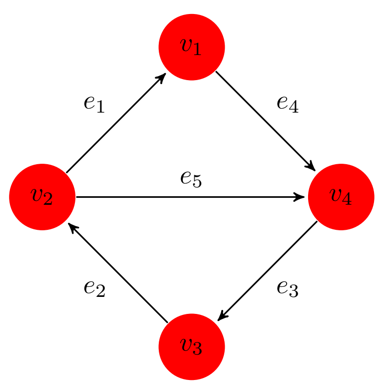

Figure 1.3. For the spanning tree T with tree edges e_1, e_2 and e_5 the element $\mathbf{s}_{P_5} = (0, 0, -1, 1, 1)$ is the fundamental cut vector belonging to e_5 and T, and $\mathbf{z}_{C_4} = (1, 0, 0, 1, -1)$ is the fundamental cycle vector belonging to e_4 and T.

Clearly $\{\mathbf{s}_{P_i}|\ 1 \leq i \leq n-1\}$ is an independent set of cut vectors, because if $\sum_{i=1}^{n-1} \lambda_i \mathbf{s}_{P_i} = 0$ then for every $1 \leq j \leq n-1$ we have that $0 = \sum_{i=1}^{n-1} \lambda_i \mathbf{s}_{P_i}(e_j) = \sum_{i=1}^{n-1} \lambda_i \delta_{ij} = \lambda_j$. By a similar argument we see that $\{\mathbf{z}_{C_i}|\ n \leq i \leq m\}$ is also a set of independent vectors which proves the claim that $\text{Dim}(Z(G)) \geq m-n+1$ and $\text{Dim}(S(G)) \geq n-1$. With the argument that for every $\mathbf{x} \in Z(G) \cap S(G)$ we know that $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Rightarrow \mathbf{x} = \mathbf{0}$, we find $\mathbb{R}^E = Z(G) \oplus S(G)$. ■

The previous proof gives us the possibility to choose a basis for $Z(G)$ and $S(G)$ by choosing a spanning tree. Since there is an orientation on the graph we know that for every e_j there exists an initial vertex and a terminal vertex.

Definition. 1.4. The *incidence matrix* B of G is the $n \times m$ matrix defined by

$$(B)_{ij} = \begin{cases} 1 \text{ if } v_i \text{ is the initial vertex of } e_j, \\ -1 \text{ if } v_i \text{ is the terminal vertex of } e_j, \\ 0 \text{ otherwise.} \end{cases}$$

We want to identify the matrix B with a linear map $B : \mathbb{R}^E \to \mathbb{R}^V$ and $B^\top$ with a linear map $B^\top : \mathbb{R}^V \to \mathbb{R}^E$. These linear maps give motivation for the following Lemma.

Lemma 1.5. *The kernel of B is equal to $Z(G)$.*

Proof. Let $\mathbf{z}_L \in Z(G)$, then we have that $(B\mathbf{z}_L)_i = \langle \mathbf{b}_i, \mathbf{z}_L \rangle$ for $\mathbf{b}_i$ the ith $(1 \le i \le n)$ row of B. Note that here for any $i = 1, 2, \ldots, n$ we have that

$$\langle \mathbf{b}_i, \mathbf{z}_L \rangle = \#\{uv \in L |\ v_i = u\} - \#\{uv \in L |\ v_i = v\} = 0.$$

We conclude that for any $\mathbf{x} \in Z(G)$ we have $B\mathbf{x} = \mathbf{0}$. Now let $\mathbf{s}_P \in S(G)$ be nonzero associated with a partition $P = \{V_1, V_2\}$ of G. Without loss of generality we assume that there exists an $e \in (V_1, V_2)$. For v_i the initial vertex of e we get

$$\langle \mathbf{b}_i, \mathbf{s}_P \rangle = \#\{uv \in (V_2, V_1)|\ v_i = u\} + \#\{uv \in (V_2, V_1)|\ v_i = v\} > 0,$$

from which the right hand side of the equation is always unequal to zero by the fact that $e \in \{uv \in (V_2, V_1)| v_i = u\}$. We conclude that for any nonzero $\mathbf{x} \in S(G)$ we have $B\mathbf{x} \neq \mathbf{0}$ which proves the claim that $\mathrm{Ker}(B) = Z(G)$. ■

Also note that by the previous result $\mathrm{Im}(B^\top) = \mathrm{Ker}(B)^\perp = Z(G)^\perp = S(G)$. The matrix $BB^\top$ is known as the Laplacian matrix of the graph. Since the kernel of the Laplacian is equal to the kernel of $B^\top$ it is easy to see that this kernel is exactly equal to the set of functions that give the same constant value to every edge. We can write such a vector as $\lambda\mathbf{1}$ where $\lambda \in \mathbb{R}$ and $\mathbf{1}$ is the $m-$component unit vector. Since the rows of $B^\top$ add up to 0 and since the columns of $B^\top$ are nonzero we have indeed that $B^\top\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \lambda\mathbf{1}$ for some scalar λ. If we denote by $\mathrm{Sp}\{\mathbf{1}\}$ the kernel of the Laplacian matrix we can look at the following short sequence.

Theorem 2. *The short sequence*

$$0 \longrightarrow Z(G) \xrightarrow{i} \mathbb{R}^E \xrightarrow{B} \mathrm{Sp}\{\mathbf{1}\}^\perp \longrightarrow 0,$$

with i the inclusion map and B the incidence matrix, is exact.

Proof. Note that by Lemma 1.5 we have that $\mathrm{Ker}(B) = Z(G) = \mathrm{Im}(i)$. Also we know that $\mathrm{Ker}(i) = \mathbf{0}$ and $\mathrm{Im}(B) = \mathrm{Ker}(B^\top)^\perp = \mathrm{Sp}\{\mathbf{1}\}^\perp$. ■

Theorem 3. (Matrix Tree Theorem) *The number of spanning trees of G is equal to* $\mathrm{Det}(\tilde{B}\tilde{B}^\top)$*, where* $\tilde{B}$ *is obtained from B by omitting the last row.*

PROOF. Remark that the cases $\#V(G) \in \{1,2\}$ are trivial and thus we assume that $\#V(G) \geq 3$. Let $\tilde{B}$ the $(n-1)\times m$ matrix obtained by omitting the last row from B. The Cauchy-Binet formula now tells us the following

$$\mathrm{Det}(\tilde{B}\tilde{B}^\top) = \textstyle\sum_J \mathrm{Det}(\tilde{B}(J))\mathrm{Det}(\tilde{B}^\top(J)) = \sum_J \mathrm{Det}(\tilde{B}(J))\mathrm{Det}(\tilde{B}(J)^\top),$$

where the summation is over all $(n-1)$-subsets J of $\{1,2,\ldots,m\}$, and $\tilde{B}(J)$ is the $(n-1)\times(n-1)$ submatrix of $\tilde{B}$ formed by the columns of $\tilde{B}$ indexed with elements of J and $\tilde{B}^\top(J)$ is the submatrix of $\tilde{B}^\top$ formed by the corresponding rows of $\tilde{B}^\top$. Let J now be a $(n-1)$-subset of $\{1,2,\ldots,m\}$.

We now have two cases to consider. The edges corresponding to the columns of $\tilde{B}(J)$ form a tree or not. Let G_J be the subgraph of G formed by these $n-1$ edges with $V(G_J) = V(G)$. If G_J is a tree we know it is a tree of $n-1$ edges and thus a spanning tree of G. We then know that G_J has at least two leaves, i.e., it has at least two vertices $v \in V(G_J)$ such that $\deg(v) = 1$. Let $v_i \in V(G_J) \subset V(G)$ be a leaf that is not equal to $v_n \in V(G)$. Then v_i corresponds to a row in $\tilde{B}(J)$ with only one nonzero entry. If we now use the cofactor expansion along this row, we see that $\mathrm{Det}(\tilde{B}(J)) = \pm 1\mathrm{Det}(\tilde{B}(J)')$, where $\tilde{B}(J)'$ is the submatrix of $\tilde{B}(J)$ without the row corresponding to the leaf and without the column corresponding to the (unique) edge which is connected to this leaf. Note that again the edges corresponding to the columns of this submatrix form a tree and we can again use the cofactor expansion along the row corresponding to a leaf not equal to v_n. Repeatedly using these cofactor expansions gives us the result $\mathrm{Det}(\tilde{B}(J)) = \prod_{i=1}^{n-1} \pm 1 = \pm 1$.

For G_J not a tree we will first show that G_J has a cycle. Suppose G_J does not have a cycle, then it is a forest, i.e., a disjoint union of trees $T_1, T_2, \ldots, T_l$ for some integer l. If T_i has n_i vertices, it has $n_i - 1$ edges. Therefore we have $n-1 = \sum_{i=1}^{l} n_i - 1 \leq n - l$, which implies that l must be equal to 1 and thus gives a contradiction with the assumption that G_J is not a tree. We conclude that G_J has a cycle. Note that if we look at the columns corresponding to the edges of this cycle, we have that the sum of these columns is equal to zero, since every cycle has as many initial edges as terminal edges in every vertex of the cycle. We conclude that the column space of $\tilde{B}(J)$ is not of maximal dimension and thus we obtain $\mathrm{Det}(\tilde{B}(J)) = 0$. The wanted result is now obtained by the fact that

$$\mathrm{Det}(\tilde{B}(J))\mathrm{Det}(\tilde{B}(J)^\top) = \mathrm{Det}(\tilde{B}(J))^2 = \begin{cases} 1 \text{ if } G_J \text{ is a tree,} \\ 0 \text{ if } G_J \text{ is not a tree,} \end{cases}$$

and the Cauchy-Binet formula. ■

Note that in the proof we can replace $\tilde{B}$ by a matrix obtained by omitting any row of B and still have the desired result as the theorem states. The previous theorem is also called Kirchhoff's theorem. Kirchhoff is very famous by his circuit laws which we are going to take a closer look at in the following section.

2 Effective resistance

2.1 A linear algebra approach

In this section we take a look at different ways to compute the effective resistance between two vertices in a graph. The main approach is to look at the Kirchhoff laws in different forms.

For the first formula for the effective resistance we are going to look at the weighted Laplacian, but first we will state Kirchhoff's laws and Ohm's law in terms of vectors. Let G be a graph as stated in the first section with $n > 1$ vertices and m edges. To compute the effective resistance between distinct vertices $v_x, v_y \in V(G)$ we let a voltage source be connected between vertices v_x and v_y and let $I > 0$ be the net current out of source v_x and into sink v_y. Since we now look at G as an electrical network we have assigned a voltage, current and resistance on each edge. These can be written in the form of vectors of our edge space. Let $\boldsymbol{\iota} \in \mathbb{R}^E$ be the *current vector*, $\boldsymbol{\upsilon} \in \mathbb{R}^E$ be the *potential vector* and $\boldsymbol{\rho} \in \mathbb{R}^E$ be the *resistance vector*. Furthermore we let $\boldsymbol{\iota}(uv) = -\boldsymbol{\iota}(vu)$, $\boldsymbol{\upsilon}(uv) = -\boldsymbol{\upsilon}(vu)$ and $\boldsymbol{\rho}(uv) = \boldsymbol{\rho}(vu) > 0$ for all $uv \in E(G)$. Kirchhoff's first law (the current law) is now stated as

$$\sum_{v \in N(u)} \boldsymbol{\iota}(uv) = \begin{cases} I \text{ if } u = v_x, \\ -I \text{ if } u = v_y, \\ 0 \text{ otherwise,} \end{cases}$$

where $N(u)$ is the neighbourhood of u, i.e., $N(u) = \{v \in V | vu \in E \text{ or } uv \in E\}$.

Kirchhoff's second law (the potential law) can be stated as saying that for every cycle L of G we have $\sum\limits_{uv \in L} \boldsymbol{\upsilon}(uv) = 0$. This is equivalent to say that we can look at a potential vector $\tilde{\boldsymbol{\upsilon}}$ as an element of $\mathbb{R}^V$ for which it holds that for all $uv \in E(G)$ that $\boldsymbol{\upsilon}(uv) = \tilde{\boldsymbol{\upsilon}}(u) - \tilde{\boldsymbol{\upsilon}}(v)$. The effective resistance between v_x and v_y is now said to be equal to $r(v_x, v_y) = I^{-1}\left(\tilde{\boldsymbol{\upsilon}}(v_x) - \tilde{\boldsymbol{\upsilon}}(v_y)\right)$, and by Ohm's law we have for all $uv \in E(G)$ that

$$\boldsymbol{\iota}(uv)\boldsymbol{\rho}(uv) = \tilde{\boldsymbol{\upsilon}}(u) - \tilde{\boldsymbol{\upsilon}}(v).$$

Definition. 2.1. The *weighted Laplacian* Q of graph G with weight vector $\mathbf{w} \in \mathbb{R}^E$, such that $\mathbf{w}(uv) = \mathbf{w}(vu)$ for all $uv \in E(G)$, is the $n \times n$ matrix given by

$$(Q)_{ij} = \begin{cases} \sum_{u \in N(v_i)} \mathbf{w}(v_i u) & \text{if } v_i = v_j, \\ -\mathbf{w}(v_i v_j) & \text{if } v_i v_j \in E(G) \text{ or } v_j v_i \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Definition. 2.2. The *Laplacian pseudoinverse* Q^+ of the weighted Laplacian Q is defined as the unique matrix satisfying $Q^+\mathbf{1} = \mathbf{0}$ and

$$\text{for all } \mathbf{u} \in \mathrm{Sp}\{\mathbf{1}\}^\perp : Q^+\mathbf{u} = \mathbf{v} \text{ if and only if } Q\mathbf{v} = \mathbf{u} \text{ and } \mathbf{v} \in \mathrm{Sp}\{\mathbf{1}\}^\perp.$$

Theorem 4. *If the weight vector* $\mathbf{w}$ *is given by* $\mathbf{w}(uv) = \frac{1}{\boldsymbol{\rho}(uv)}$ *for all* $uv \in E$ *(i.e. the edge weights refer to conductances), then the effective resistance between* $v_x, v_y \in V(G)$ *is equal to*

$$r(v_x, v_y) = (\mathbf{v}_x - \mathbf{v}_y)^\top Q^+ (\mathbf{v}_x - \mathbf{v}_y),$$

where $\mathbf{v}_x$ *(respectively* $\mathbf{v}_y$*) is the* x*th (respectively* y*th) standard basis vector of the vertex space.*

Proof. If we substitute the equation of Ohm's law into the equation obtained from the current law we have

$$\sum_{v \in N(u)} \frac{\tilde{\boldsymbol{v}}(u) - \tilde{\boldsymbol{v}}(v)}{\boldsymbol{\rho}(uv)} = \begin{cases} I \text{ if } u = v_x, \\ -I \text{ if } u = v_y, \\ 0 \text{ otherwise.} \end{cases}$$

In matrix form we can write this as

$$Q\tilde{\boldsymbol{v}} = I(\mathbf{v}_x - \mathbf{v}_y).$$

Now note that $\mathbf{v}_x - \mathbf{v}_y$ is perpendicular to $\mathbf{1}$ and thus perpendicular to the kernel of the weighted Laplacian. Now note that the kernel of the weighted Laplacian is indeed the same as that of the Laplacian of the first section, namely Sp$\{\mathbf{1}\}$. If we now look at the Laplacian pseudoinverse we have that $\tilde{\boldsymbol{v}} = IQ^+(\mathbf{v}_x - \mathbf{v}_y)$. Therefore we obtain

$$r(v_x, v_y) = \frac{\tilde{\boldsymbol{v}}(v_x) - \tilde{\boldsymbol{v}}(v_y)}{I} = \frac{(\mathbf{v}_x - \mathbf{v}_y)^\top \tilde{\boldsymbol{v}}}{I} = (\mathbf{v}_x - \mathbf{v}_y)^\top Q^+ (\mathbf{v}_x - \mathbf{v}_y),$$

which completes the proof. ■

Corollary 2.3. *If we define* $r(G) := \sum_{i=1}^n \sum_{j=i+1}^n r(v_i, v_j)$ *to be the total resistance of* G*, we obtain*

$$r(G) = n \sum_{i=2}^n \frac{1}{\lambda_i},$$

where $\lambda_2, \lambda_3, \ldots, \lambda_n$ *are the nonzero eigenvalues of* Q*.*

Proof. By Theorem 4 and the fact that $r(v_i, v_j) = r(v_j, v_i)$ for all $v_i, v_j \in V(G)$, we obtain

$$\begin{aligned} r(G) &= \sum_{i=1}^n \sum_{j=i+1}^n r(v_i, v_j) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\mathbf{v}_i - \mathbf{v}_j)^\top Q^+ (\mathbf{v}_i - \mathbf{v}_j) \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left((Q^+)_{ii} - 2(Q^+)_{ij} + (Q^+)_{jj}\right) \\ &= n \sum_{i=1}^n (Q^+)_{ii} - \mathbf{1}^\top Q^+ \mathbf{1} = n\text{Trace}(Q^+). \end{aligned}$$

Since Q is symmetric we know it has an orthogonal basis of eigenvectors. Let F be the matrix with the ith column being the eigenvector corresponding to eigenvalue λ_i (with $\lambda_1 = 0$) and let D be the diagonal matrix with the vector $(0, \lambda_2, \lambda_3, \ldots, \lambda_n)$ as diagonal. The weighted Laplacian now satisfies $Q = FDF^{-1} = FDF^\top$. So with respect to this basis of eigenvectors we have that the pseudoinverse is equal to D^+, which is the diagonal matrix with the vector $(0, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \ldots, \frac{1}{\lambda_n})$ as diagonal. Because similar matrices have the same trace we obtain

$$r(G) = n\text{Trace}(Q^+) = n\text{Trace}(D^+) = n\sum_{i=2}^{n} \frac{1}{\lambda_i}.$$

This proves the statement. ■

To give more intrinsic understanding to this theorem we will see that the above equation also will appear in a more geometric environment. This can be done by looking at orthogonal projections, but first we will have to look at the following definition.

Definition. 2.4. Let $u_1, u_k \in V$ be distinct. A *path* Γ from u_1 to u_k of G is a graph such that $V(\Gamma) = \{u_1, \ldots, u_k\} \subset V(G)$ and $E(\Gamma) = \{u_1u_2, \ldots, u_{k-1}u_k\}$ such that for every $uv \in E(\Gamma)$ we have $uv \in E(G)$ or $vu \in E(G)$.

Every path Γ of G can be identified with an element $\boldsymbol{\gamma}_\Gamma \in \mathbb{R}^E$:

$$\text{For every } uv \in E(G) \text{ we have } \boldsymbol{\gamma}_\Gamma(uv) = \begin{cases} 1 \text{ if } uv \in E(\Gamma), \\ -1 \text{ if } vu \in E(\Gamma), \\ 0 \text{ otherwise.} \end{cases}$$

Note that the definition of a cycle is now very similar to that of a path. Moreover we can also define a cycle to be a path from a vertex $v_i \in V$ to itself. Also note here that for any standard basis vector $\mathbf{e}_i$ of $\mathbb{R}^E$ we have that there exists a path Γ such that $\boldsymbol{\gamma}_\Gamma = \mathbf{e}_i$, namely the path Γ that only consists of e_i with its initial en terminal vertex. Therefore note that the subspace of $\mathbb{R}^E$ which is spanned by the elements $\boldsymbol{\gamma}_\Gamma$ as Γ runs over the set of paths is equal to $\mathbb{R}^E$ itself. This gives reason to look at the orthogonal decomposition of any path.

For the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top R\mathbf{y}$ (where R is the diagonal matrix with $\boldsymbol{\rho}$ as its diagonal) we see that $Z(G)$ and $S(G)$ are not perpendicular subspaces any more. Therefore we look at the *reformed cut space* $\{R^{-1}\mathbf{u}_P | \mathbf{u}_P \in S(G)\}$ which we denote as $\tilde{S}(G)$. Now note that $Z(G) \perp \tilde{S}(G)$ and therefore we have for any path vector $\boldsymbol{\gamma}$ by the Pythagorean theorem that

$$||\boldsymbol{\gamma}||^2 = ||P_{\tilde{S}}\boldsymbol{\gamma}||^2 + ||P_Z\boldsymbol{\gamma}||^2,$$

where P_Z (respectively $P_{\tilde{S}}$) is the orthogonal projection matrix onto $Z(G)$ (respectively $\tilde{S}(G)$). To get more understanding about these orthogonal projection matrices we take a look at the following lemma.

Lemma 2.5. *Let* $\mathbb{R}^n$ *be the* $n-$*dimensional inner product space with inner product* $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top D \boldsymbol{y}$ *for a particular invertible diagonal matrix* D *with positive entries. Furthermore let* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ *be linearly independent elements of* $\mathbb{R}^n$ *and let* A *be the* $n \times k$ *matrix whose* i*-th column is equal to* $\mathbf{a}_i$*. Then the orthogonal projector onto* $W = \mathrm{Sp}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ *with respect to the standard basis is given by*

$$P = A(A^\top DA)^{-1}A^\top D.$$

Proof. Let $\mathbf{x} \in \mathbb{R}^n$. Note that we can uniquely write $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 \in \mathbb{R}^n$ such that $\mathbf{x}_1 \in W$ and $\mathbf{x}_2 \in W^\perp$. Now there exists an $\mathbf{y} \in \mathbb{R}^k$ such that $A\mathbf{y} = \mathbf{x}_1$. We obtain that

$$P\mathbf{x}_1 = \mathbf{x}_1 = A\mathbf{y} = A(A^\top DA)^{-1}(A^\top DA)\mathbf{y} = A(A^\top DA)^{-1}A^\top D\mathbf{x}_1.$$

Now to prove that $(A^\top DA)^{-1}$ is indeed invertible let us assume the contrary. Then there exists a $\mathbf{z} \in \mathbb{R}^k$ such that $A^\top DA\mathbf{z} = \mathbf{0}$. Now let $\sqrt{D}$ be the matrix obtained by taking the square root of all entries of D, then we obtain

$$\mathbf{z}^\top A^\top DA\mathbf{z} = \mathbf{z}^\top A^\top \sqrt{D}\sqrt{D}A\mathbf{z} = (\sqrt{D}A\mathbf{z})^\top \sqrt{D}A\mathbf{z} = 0.$$

We conclude that $\sqrt{D}A\mathbf{z} = \mathbf{0}$, which contradicts with the assumption that the columns of A were linearly independent and the assumption of D being a diagonal matrix with positive entries. Furthermore we know that for any $\mathbf{w} \in W^\perp$ we have

$$\mathbf{w} \in \mathrm{Im}(A)^\perp \Rightarrow \forall \mathbf{v} \in \mathbb{R}^k : \langle \mathbf{w}, A\mathbf{v} \rangle = 0 \Rightarrow \forall \mathbf{v} \in \mathbb{R}^k : \langle A^\top \mathbf{w}, \mathbf{v} \rangle = 0,$$

which implies $\mathbf{w}$ being an element of $\mathrm{Ker}(A^\top D)$. Thus for $\mathbf{x}_2 \in W^\perp$ we have that $A(A^\top DA)^{-1}A^\top D\mathbf{x}_2 = A(A^\top DA)^{-1}\mathbf{0} = \mathbf{0}$. We finally obtain that $A(A^\top DA)^{-1}A^\top D\mathbf{x} = A(A^\top DA)^{-1}A^\top D(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{x}_1 + \mathbf{0} = \mathbf{x}_1$, which proves the statement. ■

Proposition 2.6. *Let* $\boldsymbol{\gamma}$ *be any path vector associated to a path* Γ *of* G *from* v_x *to* v_y*, then it holds that*

$$||P_{\tilde{S}}\boldsymbol{\gamma}||^2 = (\mathbf{v}_x - \mathbf{v}_y)^\top Q^+(\mathbf{v}_x - \mathbf{v}_y).$$

Proof. Lemma 2.5 gives reason to construct $P_{\tilde{S}}$ in the same way as the proof of the lemma. Let $\mathbf{x} \in \mathbb{R}^E$. Note that we can uniquely write $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 \in \mathbb{R}^E$ such that $\mathbf{x}_1 \in \tilde{S}(G)$ and $\mathbf{x}_2 \in Z(G)$. Now by the fact that our exact sequence of Theorem 2 splits, we have that $B^\top$ restricted to $\mathrm{Sp}\{\mathbf{1}\}$ gives us an isomorphism between $\mathrm{Sp}\{\mathbf{1}\}^\perp$ and S. Therefore there exists an $\mathbf{y} \in \mathrm{Sp}\{\mathbf{1}\}^\perp$ such that $R^{-1}B^\top\mathbf{y} = \mathbf{x}_1$. By the fact that $Q = BR^{-1}B^\top$, we obtain that

$$P_{\tilde{S}}\mathbf{x}_1 = \mathbf{x}_1 = R^{-1}B^\top\mathbf{y} = R^{-1}B^\top Q^+Q\mathbf{y} = R^{-1}B^\top Q^+B\mathbf{x}_1.$$

Also by the fact that $R^{-1}B^\top Q^+B\mathbf{x}_2 = R^{-1}B^\top Q^+\mathbf{0} = \mathbf{0}$ we conclude that indeed $P_{\tilde{S}} = R^{-1}B^\top Q^+B$. For any path vector $\boldsymbol{\gamma}$ as stated in the proposition it holds that

$$\begin{aligned}||P_{\tilde{S}}\boldsymbol{\gamma}||^2 &= (P_{\tilde{S}}\boldsymbol{\gamma})^\top R P_{\tilde{S}}\boldsymbol{\gamma} = \boldsymbol{\gamma}^\top P_{\tilde{S}}^\top R P_{\tilde{S}}\boldsymbol{\gamma} \\ &= \boldsymbol{\gamma}^\top B^\top Q^+ B R^{-1} R R^{-1} B^\top Q^+ B\boldsymbol{\gamma} \\ &= (B\boldsymbol{\gamma})^\top Q^+ B R^{-1} B^\top Q^+ B\boldsymbol{\gamma} \\ &= (B\boldsymbol{\gamma})^\top Q^+ Q Q^+ B\boldsymbol{\gamma} \\ &= (B\boldsymbol{\gamma})^\top Q^+ B\boldsymbol{\gamma}.\end{aligned}$$

Since we have that $\boldsymbol{\gamma}$ is the vector associated with a path from v_x to v_y we obtain

$$(B\boldsymbol{\gamma})_i = \begin{cases} 1 \text{ if } i = x, \\ -1 \text{ if } i = y, \\ 0 \text{ otherwise.} \end{cases}$$

This gives us the desired result. ∎

Because we assumed that G is connected, we know that there exists a path between every two vertices. Also we know there exists a shortest path between any two vertices, i.e., there exists a path vector $\boldsymbol{\delta}$ associated with a path from v_i to v_j such that $||\boldsymbol{\delta}||^2 \leq ||\boldsymbol{\gamma}_\Gamma||^2$ for any path Γ between v_i and v_j. Furthermore we define the distance to be $s : V(G) \times V(G) \to \mathbb{R}_{\geq 0}$, $s(u, v) = ||\boldsymbol{\delta}_{uv}||^2$, where $\boldsymbol{\delta}_{uv}$ is a shortest path vector associated with a shortest path Δ_{uv} between u and v. We see that s is indeed a metric on $V(G)$. Positive definiteness and symmetry are a direct consequence of the fact that $||\boldsymbol{\delta}_{uv}||^2 = 0 \Leftrightarrow \boldsymbol{\delta}_{uv} = \mathbf{0} \Leftrightarrow u = v$ and $||\boldsymbol{\delta}_{uv}||^2 = ||\boldsymbol{\delta}_{vu}||^2$. For the triangle inequality we have for any $w \in V(G)$ that

$$||\boldsymbol{\delta}_{uv}||^2 \leq ||\boldsymbol{\delta}_{uw} + \boldsymbol{\delta}_{wv}||^2 \leq ||\boldsymbol{\delta}_{uw}||^2 + ||\boldsymbol{\delta}_{wv}||^2,$$

since $\langle \boldsymbol{\delta}_{uw}, \boldsymbol{\delta}_{wv} \rangle = -\#\{uv \in E(G) | uv \in E(\Delta_{uw}) \text{ and } uv \in E(\Delta_{wv})\}$.

To look at the orthogonal projection matrix onto Z we first have to find $m - n + 1$ linearly independent vectors of Z. We already saw that this can be done by choosing a spanning tree and looking at the fundamental cycles associated to the chords. Let T be a spanning tree in G and again label the edges such that $e_1, e_2, \ldots, e_{n-1}$ are the tree edges and $e_n, e_{n+1}, \ldots, e_m$ are the chords. Let the *fundamental cycle matrix* C be the $m \times (m - n + 1)$ matrix whose ith column, for $1 \leq i \leq m - n + 1$, is the fundamental cycle vector $\mathbf{z}_i := \mathbf{z}_{C_{i+n-1}}$ belonging to edge e_{i+n-1} and T. At this moment we have enough tools to prove the following theorem.

Theorem 5. *Let $\Omega := C^\top RC$ be the Gramian matrix of the set of vectors $(\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_{m-n+1})$ with respect to our inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top R\boldsymbol{y}$ and let $\mathbf{v}_{xy} := C^\top R\boldsymbol{\delta}$ for $\boldsymbol{\delta}$ a shortest path vector between v_x and v_y. It then holds that*

$$r(v_x, v_y) = s(v_x, v_y) - \mathbf{v}_{xy}^\top \Omega^{-1} \mathbf{v}_{xy}.$$

Proof. Let $\boldsymbol{\delta}$ be as above. We obtain by Proposition 2.6 & Theorem 4 that

$$s(v_x, v_y) = ||\boldsymbol{\delta}||^2 = ||P_{\tilde{S}}\boldsymbol{\delta}||^2 + ||P_Z\boldsymbol{\delta}||^2 = r(v_x, v_y) + ||P_Z\boldsymbol{\delta}||^2.$$

Now by Lemma 2.5 we have

$$\begin{aligned}
||P_Z\boldsymbol{\delta}||^2 &= (C(C^\top RC)^{-1}C^\top R\boldsymbol{\delta})^\top RC(C^\top RC)^{-1}C^\top R\boldsymbol{\delta} \\
&= \boldsymbol{\delta}^\top RC(C^\top RC)^{-1}C^\top RC(C^\top RC)^{-1}C^\top R\boldsymbol{\delta} \\
&= \mathbf{v}_{xy}^\top (C^\top RC)^{-1}\mathbf{v}_{xy} \\
&= \mathbf{v}_{xy}^\top \Omega^{-1}\mathbf{v}_{xy},
\end{aligned}$$

which completes the proof. ■

Corollary 2.7. *G is a tree if and only if $r(u, v) = s(u, v)$ for all $u, v \in V(G)$.*

Proof. First assume that G is a tree. By definition G has no cycles and thus $Z(G)$ is the zero subspace. Therefore $||P_Z\boldsymbol{\delta}||^2 = 0$, which implies by the former theorem the wanted result. For the other implication assume that $r(u, v) = s(u, v)$ for all $u, v \in V(G)$. By the former theorem we know that $||P_Z\boldsymbol{\delta}||^2 = 0 \Rightarrow \boldsymbol{\delta} \in S(G)$ for every shortest path $\boldsymbol{\delta}$. Therefore it holds that $\mathbb{R}^E \subset \text{Span}\{\boldsymbol{\delta}_{uv} | uv \in E(G)\} \subset S(G)$. By the fact that also $S(G) \subset \mathbb{R}^E$ we obtain that $\mathbb{R}^E = S(G)$, which implies G is a tree. ■

To give a more physical understanding about the second formula for the effective resistance we can again look at the Kirchhoff circuit laws and Ohm's law, but this time in another fashion as we follow [4].

Let us assume that G is again the graph of our electrical network and thus we have assigned a voltage, current and resistance on each edge. This time we determine the effective resistance between two distinct vertices $v_x, v_y \in V(G)$ by connecting v_x and v_y by an additional edge e_{m+1} directed from v_x to v_y with zero resistance. We obtain a new graph G' with $V(G') = V(G)$ and $E' := E(G') = E(G) \cup \{e_{m+1}\}$. We write the current vector as $\boldsymbol{\iota}'$, the voltage vector as $\boldsymbol{\upsilon}'$ and the resistance vector as $\boldsymbol{\rho}'$. Note here that $\boldsymbol{\upsilon}', \boldsymbol{\iota}', \boldsymbol{\rho}' \in \mathbb{R}^{E'}$ with $\boldsymbol{\iota}'(e_{m+1}) = -I$, $\boldsymbol{\upsilon}'(e_{m+1}) = 1$ and $\boldsymbol{\rho}'(e_{m+1}) = 0$. In this situation we see that Kirchhoff's current law takes the form

$$B'\boldsymbol{\iota}' = \mathbf{0},$$

where B' is the incidence matrix of G'.

Furthermore let $\boldsymbol{\delta} \in \mathbb{R}^E$ be a shortest path vector associated with a path of G from v_y to v_x and let T be a spanning tree for G' which contains this path and has e_{m+1} indexed as the last chord. Now let C' be the fundamental cycle matrix for G' with respect to T. By the conditions on T we know that the $(m+1)$th column of C' is equal to $(\boldsymbol{\delta}|1)^\top$. Now Kirchhoff's voltage law can be written in the form

$$C'^{\top}\boldsymbol{\upsilon}' = \mathbf{0}.$$

Lastly we need one more equation. For this equation let R' be the $(m+1)\times(m+1)$ diagonal matrix with $\boldsymbol{\rho}'$ as its diagonal. Then Ohm's law gives us the following formula

$$\boldsymbol{\upsilon}' = R'\boldsymbol{\iota}' + \mathbf{e}_{m+1},$$

where $\mathbf{e}_{m+1}$ is the $(m+1)$th standard basis vector of $\mathbb{R}^{E'}$.

Proposition 2.8. *The current vector is given by* $\boldsymbol{\iota}' = -C'(C'^{\top}R'C')^{-1}C'^{\top}\mathbf{e}_{m+1}$.

Proof. By Kirchhoff's current law we see that $\boldsymbol{\iota}' \in Z(G)$, therefore we obtain by Lemma 2.5 and Ohm's law that

$$\boldsymbol{\iota}' = C'(C'^{\top}R'C')^{-1}C'^{\top}R'\boldsymbol{\iota}' = C'(C'^{\top}R'C')^{-1}C'^{\top}(\boldsymbol{\upsilon}' - \mathbf{e}_{m+1}).$$

Note that unlike Lemma 2.5 requires, R' does not have only positive entries. Nevertheless, $C'^{\top}R'C'$ is still invertible because $\boldsymbol{\delta} \notin Z(G)$ and thus the columns of $\sqrt{R'}C' = \left(\begin{array}{c|c} \sqrt{R}C & \sqrt{R}\boldsymbol{\delta} \\ \hline \mathbf{0} & 0 \end{array}\right)$ are still linearly independent. We conclude that by Kirchhoff's potential law we now have

$$\boldsymbol{\iota}' = C'(C'^{\top}R'C')^{-1}C'^{\top}(\boldsymbol{\upsilon}' - \mathbf{e}_{m+1}) = -C'(C'^{\top}R'C')^{-1}C'^{\top}\mathbf{e}_{m+1}.$$

This proves the statement. ■

The effective resistance between v_x and v_y is now minus the voltage on the last edge divided by the current on that edge, i.e.,

$$r(v_x, v_y) = \frac{-\boldsymbol{\upsilon}(e_{m+1})}{I} = -\frac{1}{\mathbf{e}_{m+1}^{\top}\boldsymbol{\iota}'} = \frac{1}{\mathbf{e}_{m+1}^{\top}C'(C'^{\top}R'C')^{-1}C'^{\top}\mathbf{e}_{m+1}}.$$

Note here that $C'^{\top}\mathbf{e}_{m+1} = \mathbf{e}_{m+1}$ by the way we chose C', so therefore we obtain

$$r(v_x, v_y) = \frac{1}{\mathbf{e}_{m+1}^{\top}(C'^{\top}R'C')^{-1}\mathbf{e}_{m+1}} = \left(\mathbf{e}_{m+1}^{\top}\frac{\operatorname{Adj}(C'^{\top}R'C')}{\operatorname{Det}(C'^{\top}R'C')}\mathbf{e}_{m+1}\right)^{-1}.$$

Now note here that $C'^{\top}R'C'$ can be written as

$$C'^{\top}R'C' = \begin{pmatrix} \Omega & \mathbf{v}_{xy} \\ \mathbf{v}_{xy}^{\top} & s(v_x, v_y) \end{pmatrix},$$

where Ω and $\mathbf{v}_{xy}$ are defined as in Theorem 5. We finally obtain

$$\begin{aligned} r(v_x, v_y) &= \frac{\operatorname{Det}(C'^{\top}R'C')}{\operatorname{Det}(\Omega)} \\ &= \frac{1}{\operatorname{Det}(\Omega)}\operatorname{Det}\begin{pmatrix} \Omega & \mathbf{0} \\ \mathbf{v}_{xy}^{\top} & 1 \end{pmatrix}\operatorname{Det}\begin{pmatrix} Id & \Omega^{-1}\mathbf{v}_{xy} \\ \mathbf{0} & s(v_x, v_y) - \mathbf{v}_{xy}^{\top}\Omega^{-1}\mathbf{v}_{xy} \end{pmatrix} \\ &= \frac{1}{\operatorname{Det}(\Omega)}\operatorname{Det}(\Omega)(s(v_x, v_y) - \mathbf{v}_{xy}^{\top}\Omega^{-1}\mathbf{v}_{xy}) = s(v_x, v_y) - \mathbf{v}_{xy}^{\top}\Omega^{-1}\mathbf{v}_{xy}. \end{aligned}$$

2.2 A probability theory approach

Let G be a graph as stated in the first section. We define again the weight vector $\mathbf{w}$ to be given by $\mathbf{w}(uv) = \frac{1}{\boldsymbol{\rho}(uv)} = -\frac{1}{\boldsymbol{\rho}(vu)}$ for all $uv \in E$ with $\boldsymbol{\rho} \in \mathbb{R}^V$ the resistance vector and let $\mathbf{t} \in \mathbb{R}^V$ be the vector with entries $\mathbf{t}(v_i) = \sum_{v_j \in N(v_i)} \mathbf{w}(v_i v_j)$ for all $v_i \in V$ ($\mathbf{t}(v_i)$ is said to be the total conductance out of v_i).

To look at a stochastic process on our graph G we consider a Markov chain $S = (S_n)_{n \geq 0}$ on G with $\mathbb{P}_x$ the law of S given $S_0 = v_x \in V$. The $n \times n$ transition matrix $P = (p_{ij})_{v_i, v_j \in V}$, where p_{ij} is the probability to go from v_i to v_j, is given by

$$p_{ij} = \begin{cases} \frac{\mathbf{w}(v_i v_j)}{\mathbf{t}(v_i)} & \text{for all } v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}.$$

Furthermore, the *unitary invariant distribution* $\boldsymbol{\pi} \in \mathbb{R}^V$ of S is given by

$$\boldsymbol{\pi}(v) = \frac{\mathbf{t}(v)}{\sum_{v \in V} \mathbf{t}(v)} \text{ for all } v \in V,$$

since it satisfies $\sum_{v \in V} \boldsymbol{\pi}(v) = 1$ and $\boldsymbol{\pi} = \boldsymbol{\pi} P$.

Lemma 2.9. *For all $v_i \in V$ let $\sigma_i := \min\{n \in \mathbb{Z}_{>0} | S_n = v_i\}$. Then for all distinct $v_x, v_y \in V$ the following holds*

$$\mathbb{P}_x(\sigma_y < \sigma_x) = \frac{1}{\boldsymbol{\pi}(v_x)\left(\mathbb{E}_x(\sigma_y) + \mathbb{E}_y(\sigma_x)\right)}.$$

PROOF. Let ν_{iy} be the number of visits to vertex v_i strictly in between the start of the random walk and the stop in v_y. We now have

$$\begin{aligned} \mathbb{E}_x(\nu_{iy}) &= \sum_{n=0}^{\infty} \mathbb{P}_x(S_n = v_i, \sigma_y > n) = \sum_{n=0}^{\infty} \mathbb{P}_x(S_{n+1} = v_i, \sigma_y > n) \\ &= \sum_{n=0}^{\infty} \sum_{v_j \in V} \mathbb{P}_x(S_n = v_j, S_{n+1} = v_i, \sigma_y > n) \\ &= \sum_{n=0}^{\infty} \sum_{v_j \in V} \mathbb{P}_x(S_n = v_j, \sigma_y > n) p_{ji} = \sum_{v_j \in V} \mathbb{E}_x(\nu_{jy}) p_{ji}, \end{aligned}$$

Note that $\mathbb{E}_x(\nu_{iy})$ and thus similarly $\mathbb{E}_x(\nu_{iy}) + \mathbb{E}_y(\nu_{ix})$ satisfies the invariant distribution property (see [1,§2.2] for more details). Therefore we obtain

$$\mathbb{E}_x(\nu_{iy}) + \mathbb{E}_y(\nu_{ix}) = \left(\sum_{v_i \in V} \mathbb{E}_x(\nu_{iy}) + \mathbb{E}_y(\nu_{ix})\right) \boldsymbol{\pi},$$

by the fact that $\sum_{v_i \in V} \mathbb{E}_x(\nu_{iy}) = \mathbb{E}_x(\sigma_y)$ (similarly $\sum_{v_i \in V} \mathbb{E}_y(\nu_{ix}) = \mathbb{E}_y(\sigma_x)$) and because ν_{xy} has the geometric distribution with probability of failure $p = \mathbb{P}_x(\sigma_y < \sigma_x)$ with $k \in \mathbb{Z}_{>0}$ failures. We obtain

$$\frac{1}{\mathbb{P}_x(\sigma_y < \sigma_x)} = \frac{1}{p} = \mathbb{E}_x(\nu_{xy}) = \mathbb{E}_x(\nu_{xy}) + \mathbb{E}_y(\nu_{xx}) = \boldsymbol{\pi}(v_x)\left(\mathbb{E}_x(\sigma_y) + \mathbb{E}_y(\sigma_x)\right),$$

which proves the statement. ∎

Theorem 6. *It holds for all distinct $v_x, v_y \in V$ that*

$$r(v_x, v_y) = \frac{\mathbb{E}_x(\sigma_y) + \mathbb{E}_y(\sigma_x)}{\sum_{v\in V} \mathbf{t}(v)}.$$

Proof. For all $v_i \in V$ let $\tau_i := \inf\{n \in \mathbb{Z}_{\geq 0} | S_n = v_i\}$ and $p_v := \mathbb{P}_v(\tau_x < \tau_y)$. We let the voltage source be connected between vertices v_x and v_y and let $I > 0$ be the net current out of source v_x and into sink v_y. Let the voltage vector $\tilde{\boldsymbol{v}}$ be given by $\tilde{\boldsymbol{v}}(v_i) = p_i$ for all $v_i \in V$ and let $I = \mathbf{t}(v_x)p_x$. Let $\boldsymbol{\iota} \in \mathbb{R}^E$ again be our current vector and set $\boldsymbol{\iota}(uv) = -\boldsymbol{\iota}(vu)$ for all $vu \in E$. Now we know by Ohm's law that

$$\sum_{v\in N(u)} \boldsymbol{\iota}(uv) = \sum_{v\in N(u)} \mathbf{w}(uv)(\tilde{\boldsymbol{v}}(u) - \tilde{\boldsymbol{v}}(v)) = \sum_{v\in N(u)} \mathbf{w}(uv)(p_u - p_v).$$

Furthermore we have for all $u \in V$ unequal to v_x and v_y the following

$$\begin{aligned}
\sum_{v\in N(x)} \boldsymbol{\iota}(xv) &= \sum_{v\in N(x)} \mathbf{w}(xv)(p_x - p_v) = \sum_{v\in N(x)} \mathbf{w}(uv)\mathbb{P}_v(\tau_y < \tau_x) \\
&= \mathbf{t}(x) \sum_{v\in V} p_{xv}\mathbb{P}_v(\tau_y < \tau_x) = \mathbf{t}(x)\mathbb{P}_x(\sigma_y < \sigma_x) = I, \\
\sum_{v\in N(u)} \boldsymbol{\iota}(uv) &= \sum_{v\in N(u)} \mathbf{w}(uv)(p_u - p_v) = \mathbf{t}(u)p_u - \sum_{v\in N(u)} \mathbf{w}(uv)p_v \\
&= \mathbf{t}(u)p_u - \mathbf{t}(u) \sum_{v\in V} p_{uv}p_v = \mathbf{t}(u)p_u - \mathbf{t}(u)p_u = 0, \\
\sum_{v\in N(y)} \boldsymbol{\iota}(uv) &= \sum_{v\in N(y)} \mathbf{w}(yv)(p_y - p_v) = -\sum_{v\in N(y)} \mathbf{w}(yv)p_v \\
&= -\mathbf{t}(y) \sum_{v\in V} p_{yv}p_v = -\mathbf{t}(y)\mathbb{P}_y(\sigma_x < \sigma_y) \\
&= -\mathbf{t}(x)\mathbb{P}_x(\sigma_y < \sigma_x) = -I,
\end{aligned}$$

Where the second last equality follows from Lemma 2.9. Note that in this context $\tilde{\boldsymbol{v}}$ and $\boldsymbol{\iota}$ satisfy Kirchhoff's laws and Ohm's law. We conclude that

$$r(v_x, v_y) = \frac{\tilde{\boldsymbol{v}}(x) - \tilde{\boldsymbol{v}}(y)}{I} = \frac{1}{\mathbf{t}(x)\mathbb{P}_x(\sigma_y < \sigma_x)}.$$

The theorem is now a direct result of Lemma 2.9. ■

A question that can be asked is how does one come up with the idea to look at the vector $\tilde{\boldsymbol{v}}$ in the way we defined it. To give more perspective and understanding in why this works we take a closer look at the *normalized Laplacian*.

Definition. 2.10. The *normalized Laplacian* Q^N is the $n \times n$ matrix given by

$$(Q^N)_{ij} = \begin{cases} 1 & \text{if } v_i = v_j, \\ -p_{ij} & \text{otherwise.} \end{cases}$$

Lemma 2.11. *Let $\mathbf{v} \in \mathbb{R}^V$ be a harmonic vector on $V \backslash \{v_x, v_y\}$, i.e., $Q^N \mathbf{v}(v_i) = 0$ for all $v_i \in V \backslash \{v_x, v_y\}$, with $v_x \neq v_y$* and $\mathbf{v}(v_x) > \mathbf{v}(v_y)$. *Then* $\mathbf{v}$ *attains its maximal value at v_x and its minimal value at v_y*

Proof. Let $\mathbf{v} \in \mathbb{R}^V$ be a harmonic vector on $V \backslash \{v_x, v_y\}$ with $\mathbf{v}(v_x) > \mathbf{v}(v_y)$ and put $M = \max_{v_i \in V}(\mathbf{v}(v_i))$. Since $\mathbf{v}$ is bounded, the maximum is attained at some $u \in V$. Suppose that $u \notin V \backslash \{v_x, v_y\}$. By the fact that $\mathbf{v}$ is a harmonic function we obtain $Q^N \mathbf{v}(u) = \sum_{v_i \in N(u)} p_{uy}(\mathbf{v}(v_i) - \mathbf{v}(u)) = 0$. Since $p_{uv_i} > 0$ for every $v_i \in N(u)$ we deduce that $\mathbf{v} = M\mathbf{1}$ by iteration. This contradicts with the condition that $\mathbf{v}(v_x) > \mathbf{v}(v_y)$, so we know that $u \in \{v_x, v_y\}$. The same argument shows that $\min_{v_i \in V}(\mathbf{v}(v_i))$ is attained at $\{v_x, v_y\}$, since $\mathbf{v}(v_x) > \mathbf{v}(v_y)$ we proved the statement. ■

Lemma 2.12. *Let $\mathbf{v}, \mathbf{v}' \in \mathbb{R}^V$ be two harmonic vectors on $V \backslash \{v_x, v_y\}$ with $v_x \neq v_y$ such that $\mathbf{v}(v_i) = \mathbf{v}'(v_i)$ for all $v_i \in \{v_x, v_y\}$. Then $\mathbf{v} = \mathbf{v}'$.*

Proof. Let $\mathbf{v}, \mathbf{v}' \in \mathbb{R}^V$ be two harmonic vectors on $V \backslash \{v_x, v_y\}$ such that $\mathbf{v}(v_i) = \mathbf{v}'(v_i)$ for all $v_i \in \{v_x, v_y\}$. Put $\mathbf{w} = \mathbf{v} - \mathbf{v}'$. Then we know that $Q^N \mathbf{w} = Q^N \mathbf{v} - Q^N \mathbf{v}'$. So $\mathbf{w}$ is also a harmonic vector with $\mathbf{w}(v_x) = \mathbf{w}(v_y) = 0$. By the previous lemma we thus obtain that $\mathbf{w}$ attains both its maximum and its minimum at $\{v_x, v_y\}$. The former ensures that $\mathbf{w} = \mathbf{0}$. ■

Theorem 7. *We let the voltage source of* 1 *volt be connected between vertices v_x and v_y. Let $\tilde{\boldsymbol{v}} \in \mathbb{R}^V$ be the potential vector of our electrical network G and let $\boldsymbol{p} \in \mathbb{R}^V$ be the vector given by $\boldsymbol{p}(v) = \mathbb{P}_v(\tau_x < \tau_y)$ for all $v \in V$. We then obtain*

$$\tilde{\boldsymbol{v}} = \mathbf{p}.$$

Proof. For every vertex $u \in V \backslash \{v_x, v_y\}$ the current law states

$$\begin{aligned}\sum_{v \in N(u)} \boldsymbol{\iota}(uv) &= \sum_{v \in N(u)} \mathbf{w}(uv)(\tilde{\boldsymbol{v}}(u) - \tilde{\boldsymbol{v}}(v)) \\ &= \mathbf{t}(u)\tilde{\boldsymbol{v}}(u) - \sum_{v \in N(u)} \mathbf{w}(uv)\tilde{\boldsymbol{v}}(v) = 0.\end{aligned}$$

Multiplying both sides by $\frac{1}{\mathbf{t}(u)}$ gives me the equation $\tilde{\boldsymbol{v}}(u) = \sum_{v \in N(u)} p_{uv} \tilde{\boldsymbol{v}}(v)$. It now holds that $\tilde{\boldsymbol{v}}(v_x) = 1$, $\tilde{\boldsymbol{v}}(v_y) = 0$ and $Q^N \tilde{\boldsymbol{v}}(v) = 0$ for all $v \in V \backslash \{v_x, v_y\}$. For the vector $\mathbf{p}$ we already know that $\mathbf{p}(v_x) = \mathbb{P}_x = 1$ and $\mathbf{p}(v_y) = \mathbb{P}_y = 0$. We also we have by total probability that

$$\mathbf{p}(u) = \textstyle\sum_{v \in N(u)} p_{uv} \mathbf{p}(v).$$

We see that $\mathbf{p}(v_x) = 1, \mathbf{p}(v_y) = 0$ and $Q^N \mathbf{p}(v) = 0$ for all $v \in V \backslash \{v_x, v_y\}$. We conclude that by Lemma 2.11 & Lemma 2.12 the result follows. ■

3 Harmonic forms on graphs

It is well-known that a finite graph can be viewed, in many respects, as a discrete analogue of a (compact) Riemann surface. In this section we will take a look at an analogy of this kind that can be made about the things covered in section 2.1. Also we will talk a little more about this analogy, but first we take a look at a few definitions.

Definition. 3.1. A $n-$dimensional *complex manifold* X is a Hausdorff, connected, topological space with a countable base together with a covering of X by a family of open sets $\{U_\alpha\}$ and homeomorphisms $\phi_\alpha : U_\alpha \to V_\alpha$, where $V_\alpha \subset \mathbb{C}^n$ is some open set, for which for all pairs α, β the map $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is holomorphic.

A $1-$dimensional complex manifold is called a *Riemann surface*. Examples of Riemann surfaces are the complex plane $\mathbb{C}$ itself, a less trivial example of a compact Riemann surface is the one point compactification of the complex plane, which gives you the complex sphere.

To give a connection with the former sections, we can look at a discrete analogue of the compact Riemann surface.

Definition. 3.2. A *metric graph* is a compact, connected metric space (Γ, d) such that for every $x \in \Gamma$ there exists an $n \in \mathbb{Z}_{>0}$, an $\epsilon \in \mathbb{R}_{>0}$ and an open neighbourhood N_x of x that is isometric with the *n-star*

$$S(n, \epsilon) = \{z \in \mathbb{C} |\ \exists t \in [0, \epsilon) \text{ and } k \in \mathbb{Z} \text{ such that } z = te^{\frac{2\pi i k}{n}}\},$$

given by the shortest-path metric. We also consider a metric space which consists of 1 point to be a metric graph.

Note that for every $x \in \Gamma$ the element $n \in \mathbb{Z}_{>0}$ is uniquely fixed (unless x is the unique point of the $1-$point metric graph). This number $\text{val}(x)$ we call the *valence* of x. Let V_0 be the set of all $x \in \Gamma$ with valence of x not equal to 2. Note that the points $x \in \Gamma$ with $\text{val}(x) \neq 2$ are precisely those where Γ fails to look locally like an open interval. Now if we let a small neighbourhood of x be a neighbourhood N_x of $x \in V_0$ such that $y \notin N'_x$ for all $y \in V_0 \backslash \{x\}$, we see that the set $\mathcal{U}$ of these "open intervals" and "small neighbourhoods" is an open cover of Γ. By compactness there exists a finite subcover of $\mathcal{U}$, which ensures that $\#V_0 < \infty$.

To every metric space Γ we can associate a *weighted graph* G as in the former sections (but undirected) by choosing a vertex set $V(\Gamma)$ which has the following properties:

(1) $V_0 \subset V(\Gamma) < \infty$, i.e., $\Gamma \backslash V(\Gamma)$ is a finite, disjoint union of subspaces U_i isometric to open intervals.

(2) For every U_i the closure $e_i := \overline{U}_i$ is isometric to a line segment (instead of a circle) and are called the *segments* of Γ.

(3) For every $i \neq j$ we have $e_i \cap e_j = \emptyset$ or $e_i \cap e_j = \{p\}$ for p an endpoint of both e_i and e_j.

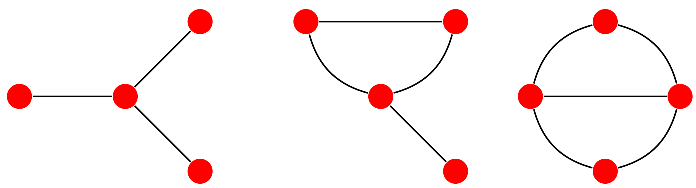

Figure 3.3. Three examples of metric graphs with vertex sets given by the red dots.

Now define G with vertices indexed by $V(\Gamma)$, and if there exists a line segment with endpoints $x, y \in \Gamma$ we join them together with an edge. Moreover, define the length of each edge as the length of the line segment, which it corresponds to. Finally G is again a weighted graph, with weights given by the reciprocals of the lengths. We call G a *model* for Γ. Note that our graph in Figure 1.3 without orientation is a model for our right metric graph in Figure 3.3 by its vertex set.

Also note that there always exists a vertex set of Γ, where the second and the third property is there to not have any loops or multiple edges in our model G. An issue that comes with these properties is that we have an infinite amount of options for our vertex set Γ, which all give another model G. This gives reason to look at the equivalence relation $\sim$ on the collection of weighted graphs. Write $G \sim G'$ if the two weighted graphs G, G' admit a *common refinement*, where we *refine* a weighted graph by subdividing its edges in a manner that preserves total length. One can check that two weighted graphs are equivalent if and only if they give rise to isometric metric graphs. This correspondence is extensively treated by Matthew Baker and Xander Faber [2], so we won't go further into it here.

First we will define a new vector space, which is more in the context of differential geometry (such as Riemann surfaces). Later on we will see an important identification of this vector space.

Let a *1-form* on G (with orientation) be an element of the real vector space with formal basis $\{\mathbf{de} : e \in E(G)\}$ and let $E^-(v)$ (respectively $E^+(v)$) be the set of all edges $e \in E(G)$ for which v is an initial vertex (respectively terminal vertex).

Definition. 3.4. For scalars $\lambda_e \in \mathbb{R}$ we call a $1-$form $\boldsymbol{\omega} = \sum \lambda_e \mathbf{de}$ on G *harmonic* if for all $v \in V(G)$ we have

$$\sum_{e \in E^+(v)} \lambda_e = \sum_{e \in E^-(v)} \lambda_e.$$

Write $\Omega(G)$ as the set of harmonic one forms on G. Since we defined forms, we want to look at integration on the elements of $\Omega(G)$. So define integration of the basic $1-$form $\mathbf{de}$ along an edge e' by

$$\int_{e'} \mathbf{de} = \begin{cases} \boldsymbol{\rho}(e) & \text{if } e = e', \\ 0 & \text{if } e \neq e'. \end{cases}$$

where the resistance vector $\boldsymbol{\rho} \in \mathbb{R}^E$ is again given by $\boldsymbol{\rho}(e) = \frac{1}{\mathbf{w}(e)} > 0$ for all $e \in E(G)$. Note that by linearity of our integral we can look at the map

$$\Omega \times \mathbb{R}^E \to \mathbb{R},\ (\boldsymbol{\omega}, \mathbf{x}) = \int_{\mathbf{x}} \boldsymbol{\omega}.$$

Lemma 3.5. *Integration restricted to $\Omega(G) \times Z(G)$ gives a perfect pairing, i.e., an isomorphism $Z(G) \xrightarrow{\sim} \Omega(G)^*$.*

PROOF. By definition we have that for any harmonic $1-$form $\boldsymbol{\omega} = \sum \lambda_e \mathbf{de}$ that $\mathbf{z} = \sum \lambda_e \mathbf{e} \in Z(G)$. In fact we have that

$$\int_{\mathbf{z}} \boldsymbol{\omega} = \sum \lambda_e^2 \boldsymbol{\rho}(e) = 0 \text{ if and only if } \lambda_e = 0 \text{ for all } e \in E(G).$$

Now let $\mathbf{y} \in \tilde{S}(G)$, we can write $\mathbf{y} = R^{-1}B^\top \mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^V$. We obtain for any harmonic $1-$form $\boldsymbol{\omega} = \sum \lambda_{uv} \mathbf{duv}$ that

$$\begin{aligned} \int_{R^{-1}B^\top \mathbf{x}} \boldsymbol{\omega} &= \sum_{uv \in E} \frac{\mathbf{x}(v) - \mathbf{x}(u)}{\boldsymbol{\rho}(uv)} \int_{uv} \boldsymbol{\omega} = \sum_{uv \in E} \frac{\mathbf{x}(v) - \mathbf{x}(u)}{\boldsymbol{\rho}(uv)} \left(\sum_{e \in E} \lambda_e \int_{uv} \mathbf{de} \right) \\ &= \sum_{uv \in E} \lambda_{uv} \left(\mathbf{x}(v) - \mathbf{x}(u) \right) = \sum_{v \in V} \mathbf{x}(v) \left(\sum_{e \in E^+(v)} \lambda_e - \sum_{e \in E^-(v)} \lambda_e \right), \end{aligned}$$

where the right hand side vanishes by the fact that $\boldsymbol{\omega}$ was harmonic. ■

Now since we have made the identification between $Z(G)$ and $\Omega(G)^*$, we again choose a spanning tree and let $(\mathbf{z}_1, ..., \mathbf{z}_g)$ be a basis of fundamental cycles for $g = m - n + 1$. Now let $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots, \boldsymbol{\omega}_g)$ be the dual basis of $(\mathbf{z}_1, ..., \mathbf{z}_g)$ determined by the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top R \mathbf{y}$. In this fashion the Gramian matrix $(\Omega)_{ij} = \langle \mathbf{z}_i, \mathbf{z}_j \rangle = \int_{\mathbf{z}_i} \boldsymbol{\omega}_j$ is now also known as the period matrix, which also has an analogue in the field of Riemann surfaces. If we now consider our vector $\mathbf{v}_{xy} = C^\top R \boldsymbol{\delta}$ from THEOREM 5, we have by the previous lemma that $(\mathbf{v}_{xy})_i = (C^\top R \boldsymbol{\delta})_i = \mathbf{z}_i^\top R \boldsymbol{\delta} = \langle \mathbf{z}_i, \boldsymbol{\delta} \rangle = \int_{\boldsymbol{\delta}} \boldsymbol{\omega}_i$, which gives the following theorem as a direct result.

Theorem 8. *Let $v_x, v_y \in V(G)$ and let $\boldsymbol{\delta}$ be a path vector associated with a path from v_x to v_y. Then the identity*

$$\mathbf{v}_{xy} = (\int_{\boldsymbol{\delta}} \boldsymbol{\omega}_1, \int_{\boldsymbol{\delta}} \boldsymbol{\omega}_2, \ldots, \int_{\boldsymbol{\delta}} \boldsymbol{\omega}_g)$$

holds. ■

4 References

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