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The structure of squarefree groups and the Schreier condition

Bachelor thesis

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Contents

1 Introduction

In this thesis we study groups. We assume the reader is familiar with the basics of group theory.

We call a positive integer *squarefree* if it is not divisible by the square of any prime number. We say a group is of *squarefree order* if it is finite and the order is squarefree. The *commutator subgroup* $[G, G]$ of G is the subgroup that is generated by all commutators of G. For $n \in \mathbb{Z}_{>0}$ we denote $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$ with $G^{(0)} := G$. A group G is called *solvable* if there exists some $m \in \mathbb{Z}_{\geq 0}$ such that $G^{(m)} = \{1\}$. It turns out that every group of squarefree order is solvable. We will see this in chapter 3.

We call a group *simple* if it has precisely two normal subgroups. A *composition series* of G is a series $\{1\} = G_t \lhd G_{t-1} \lhd \cdots \lhd G_1 \lhd G_0 = G$ such that for all $0 \leq i < t$ with $t \in \mathbb{Z}_{\geq 0}$ the *composition factors* G_i/G_{i+1} are simple. In chapter 4 we will prove that any group has only finitely many composition series.

A group is called *squarefree* if it admits a composition series and for some composition series the composition factors are pairwise non-isomorphic. It turns out that this is independent of the choice of composition series. This is a result of the Jordan-Hölder theorem which we will prove in chapter 2. For a group G that admits a composition series one can think of G as being "built up" from its composition factors. If G is squarefree, then each composition factor "occurs" at most once in the composition series of G.

The following theorem asserts a relation between groups of squarefree order and squarefree groups.

Theorem 1.1. *A group G is of squarefree order if and only if G is squarefree and solvable.*

In chapter 5 we will give a short proof of this theorem. For a group G a subgroup H is called a *subnormal subgroup* of G if there exists a series $H = H_r \lhd H_{r-1} \lhd \cdots \lhd H_1 \lhd H_0 = G$ with $r \in \mathbb{Z}_{\geq 0}$. We call H a *characteristic subgroup* of G if for every $\varphi \in \mathrm{Aut}(G)$ it holds that $\varphi(H) = H$. A surprising result on squarefree groups is the following.

Theorem 1.2. *Let G be a squarefree group and let H be a subgroup of G. Then the following are equivalent:*

1. *the subgroup H of G is subnormal in G;*
2. *the subgroup H of G is normal in G;*
3. *the subgroup H of G is characteristic in G.*

We will prove this theorem in chapter 5. A group G is called a *Schreier group* if G has a composition series and for every composition factor S of G the group $\mathrm{Out}(S)$ is solvable. We often refer to this property as the *Schreier property* or *Schreier condition*. It turns out that every finite group is a Schreier group. This is a consequence of the following theorem.

Theorem 1.3 (Schreier's Conjecture)**.** *Let G be a finite simple group. Then* $\mathrm{Out}(G)$ *is solvable.*

This conjecture was stated by Otto Schreier in 1926 and is true as a result of *the classification of finite simple groups*. We will discuss this later in chapter 6.

For squarefree Schreier groups we will prove the following analogue.

Theorem 1.4. *Let G be a Schreier group. Then the following are equivalent:*

1. *the group G is squarefree;*
2. *the quotient groups G/G', G'/G'' and $\mathrm{Z}(G'')$ are cyclic with squarefree and pairwise coprime orders and in addition $G''/\mathrm{Z}(G'')$ is the direct product of finitely many pairwise non-isomorphic non-abelian simple groups.*

In chapter 6 we will prove this theorem.

In chapter 7 we will show the reader a way to construct a finite squarefree group G such that G/G', G'/G'', $\mathrm{Z}(G'')$ and $G''/\mathrm{Z}(G'')$ are non-trivial. We assume the reader is familiar with linear groups and the theory of semidirect products. For an abelian group G and $n \in \mathbb{Z}$ we recall that the *n-torsion group* is the group denoted by $G[n] := \{g \in G : g^n = 1\}$.

Theorem 1.5. *Let $n > 1$ and $q = p^d$ for a prime p and $d \in \mathbb{Z}_{>0}$. Let $a, b, c \in \mathbb{Z}_{>0}$ and assume that $a \mid d$, $b \mid q-1$ and $c \mid \gcd(n, q-1)$. We now define*

$$\Lambda(b, n, q) := \{x \in \mathrm{GL}(n, \mathbb{F}_q) : \det x \in \mathbb{F}_q^*[b]\}.$$

Then for $\kappa(c, n, q) := \gcd(n, q-1)/c$ it follows that $G := \Lambda(b, n, q)/\mathbb{F}_q^*[\kappa(c, n, q)] \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle$ is a group and we derive the following properties:

1. *for $q > 3$ and $\gcd(p^{\frac{d}{a}} - 1, b) = 1$ it follows that $G' = \Lambda(b, n, q)/\mathbb{F}_q^*[\kappa(c, n, q)]$;*
2. *for $q > 3$ it follows that $(\Lambda(b, n, q)/\mathbb{F}_q^*[\kappa(c, n, q)])' = \mathrm{SL}(n, \mathbb{F}_q)/\mathbb{F}_q^*[\kappa(c, n, q)]$;*
3. *for $q > 3$ it follows that $\mathrm{Z}(\mathrm{SL}(n, \mathbb{F}_q)/\mathbb{F}_q^*[\kappa(c, n, q)]) = \mathbb{F}_q^*[n]/\mathbb{F}_q^*[\kappa(c, n, q)]$;*
4. *for $q > 3$, $\gcd(p^{\frac{d}{a}} - 1, b) = 1$ and a, b and c squarefree and pairwise coprime it follows that G is a squarefree group and $\#G = \frac{abc}{\gcd(n,q-1)} q^{\frac{n(n-1)}{2}} \prod_{i=2}^{n} (q^i - 1)$.*

Finally, we will show how to construct a squarefree group that does not satisfy the Schreier condition. As a consequence of the Schreier conjecture these groups must be of infinite order.

Theorem 1.6. *Let S be a non-abelian simple group and for an index set I let $k = \mathbb{Q}(X_i : i \in I)$. We assume that $\#S \leq \#I$ and S is not isomorphic to $\mathrm{PSL}(2, k)$. Then there exists a group homomorphism $\rho : S \to \mathrm{Aut}(\mathrm{PSL}(2, k))$ such that $G := \mathrm{PSL}(2, k) \rtimes_\rho S$ is a squarefree group but G does not satisfy the Schreier property and furthermore G does not satisfy (2) of Theorem 1.4.*

We will prove this theorem in chapter 7. It turns out that for each $t \in \mathbb{Z}_{>1}$ there exists a group

$$\Omega_t := \mathop{\rtimes}_{i=t}^{1} \mathrm{PSL}(2, k_i)$$

that satisfies the properties of Theorem 1.6 and has "length" t. We will properly define Ω_t at the end of chapter 7.

2 A review of group theory

Definition 2.1. The *commutator subgroup* of G is the subgroup generated by all commutators of G and is denoted by $[G, G]$ or G'. For $n \in \mathbb{Z}_{>0}$ we denote $G^{(n)} := [G^{(n-1)}, G^{(n-1)}]$ with the convention that $G^{(0)} := G$.

Definition 2.2. Let G be a group. Then G is called *perfect* if $G = G'$.

Definition 2.3. Let M and N be groups. If $[M, N] := \langle mnm^{-1}n^{-1} : m \in M, n \in N\rangle = \{1\}$, then we say that M and N *centralize each other.*

Lemma 2.4. *Let G be a group and let M and N be normal subgroups of G such that $M \cap N = \{1\}$. Then $[M, N] = \{1\}$ and $M \times N \cong MN$.*

Proof. We leave this proof as an exercise for the reader. □

Definition 2.5. A *normal series* of G is a series of subgroups

$$\{1\} = G_t \subset G_{t-1} \subset \ldots \subset G_0 = G$$

such that $G_{i+1} \lhd G_i$ for all $0 \leq i < t$ with $t \in \mathbb{Z}_{\geq 0}$.

Definition 2.6. A group is *solvable* if it has a normal series

$$\{1\} = G_t \subset G_{t-1} \subset \ldots \subset G_0 = G$$

such that the factor groups G_i/G_{i+1} are abelian for $0 \leq i < t$.

A good exercise is to show that this is equivalent to the existence of $m \in \mathbb{Z}_{\geq 0}$ such that $G^{(m)} = \{1\}$.

Theorem 2.7. *Let G and H be groups such that G is perfect and H is solvable. Then* $\mathrm{Hom}(G, H) = \{1\}$.

Proof. Let $\varphi : G \to H$ be a homomorphism. Then it follows that

$$\varphi(G) = \varphi\left(G^{(n)}\right) = \varphi(G)^{(n)} \subset H^{(n)} = \{1\}$$

for certain $n \in \mathbb{Z}_{>0}$. □

The following theorem turns out to be useful to prove that a group is solvable.

Theorem 2.8. *Let G be a group and N a normal subgroup of G. Then G is solvable if and only if N and G/N are solvable.*

The proof can be found here [4, p. 19].

Definition 2.9. A subgroup H of a group G is a *subnormal subgroup* of G if there exists a series

$$H = H_t \lhd H_{t-1} \lhd \cdots \lhd H_1 \lhd H_0 = G.$$

Definition 2.10. Let H be a subgroup of G. Then we call H a *characteristic subgroup* of G if for every $\varphi \in \mathrm{Aut}(G)$ it holds that $\varphi(H) = H$.

We leave it as an exercise to show that the property of being a characteristic subgroup is transitive. Another observation is that this is not in general true for normal subgroups. Furthermore for N a normal subgroup of G and K a characteristic subgroup of N it follows that K is a normal subgroup of G. The proof follows directly from the observation that N is invariant under inner automorphisms of G, i.e. for every $\psi \in \mathrm{Inn}(G)$ it holds that $\psi(N) = N$.

Definition 2.11. Let G be a group. Then G is called *simple* if $\#\{N : N \lhd G\} = 2$.

Note that for an abelian group G the following is equivalent: G simple if and only if G is cyclic of prime order.

Theorem 2.12. *Let G be a group and let M and N, $M \neq N$ be normal subgroups of G such that G/M and G/N are simple. Then $G/(M \cap N) \cong G/M \times G/N$.*

Proof. We will first show that $G = MN$. We note that MN is normal in G therefore MN/M is normal in G/M. By assumption we have that G/M is simple therefore $MN/M = \{1\}$ or $MN/M = G/M$. It follows that

$$MN/M = \{1\} \iff MN = M \iff N \subset M.$$

Similarly, since G/N is simple it follows that $MN/N = \{1\}$ or $MN/N = G/N$. The first case implies $M \subset N$. This is a contradiction. We conclude that $G = MN$. We now define $\varphi : G \to (G/M) \times (G/N),\ g \mapsto (gM, gN)$. Clearly φ is a homomorphism and $\ker(\varphi) = M \cap N$. Since $MN = G$ it follows that

$$\varphi(M) = \{1\} \times G/N \text{ and } \varphi(N) = G/M \times \{1\}.$$

It now follows that $\mathrm{im}(\varphi) = G/M \times G/N$ hence φ is surjective. By the first isomorphism theorem the result now follows. □

2.1 Composition series

Definition 2.13. A *composition series* of G is a normal series

$$\{1\} = G_t \subset G_{t-1} \subset \ldots \subset G_0 = G$$

such that the *composition factors* G_i/G_{i+1} are simple for all $0 \leq i < t$ with $t \in \mathbb{Z}_{\geq 0}$.

Clearly every finite group has a composition series. It turns out that not every group of infinite order has a composition series. A good exercise is to show that every abelian group with a composition series must be finite. One way to construct a group of infinite order with a composition series is to take the direct product of finitely many non-abelian simple groups of infinite order.

A composition series need not be unique. For example the cyclic group of order 12 has three composition series. Furthermore in chapter 4 we will see that any group G has only finitely many composition series.

In fact much more can be said about groups with composition series. Before we can state some nice properties we need two definitions.

Definition 2.14. Let G be a group and let

$$\{1\} = G_n \subset G_{n-1} \subset \cdots \subset G_1 \subset G_0 = G,$$
$$\{1\} = H_m \subset H_{m-1} \subset \cdots \subset H_1 \subset H_0 = G$$

be two normal series of G. We call these series *equivalent* if $n = m$ and the sequences of the factor groups are the same up to isomorphisms and permutation with $m, n \in \mathbb{Z}_{\geq 0}$.

Definition 2.15. Let G be a group. A *refinement* of a normal series

$$\{1\} = G_t \subset G_{t-1} \subset \cdots \subset G_1 \subset G_0 = G$$

is a normal series which can be obtained by inserting finitely many subgroups in the given series. We call a refinement *proper* if there exists a subgroup in the new series that is not equal to any subgroup in the original series.

We leave it as an exercise to show that a composition series admits no proper refinements. We will now state two classical theorems about composition series.

Theorem 2.16 (Schreier refinement theorem)**.** *Let G be a group. Then any two normal series of G have equivalent refinements.*

The proof of this theorem can be found here [4, p. 22]. The following theorem is a direct result of the previous theorem.

Theorem 2.17 (Jordan-Hölder)**.** *Let G be a group. Then all composition series of G are equivalent.*

Proof. If G does not have a composition series we are immediately done. Now suppose that G has a composition series. Since a composition series of G admits no proper refinements the result follows directly from Theorem 2.16. □

Lemma 2.18. *Let G be a group and $N \subset G$ a subnormal subgroup. If G has a composition series then there exists a composition series containing N.*

Proof. Let N be a subnormal subgroup of G. Then there exists a series

$$\{1\} \lhd N \lhd N_{t-1} \lhd \cdots \lhd N_1 \lhd N_0 = G$$

of G. Now since a composition series admits no proper refinements we have by the Schreier refinement theorem that some refinement of the series above must be equivalent to a composition series of G. Hence some refinement of the series above is also a composition series and contains N. □

2.2 Semidirect products

Definition 2.19. Let N and H be groups and $\sigma : H \to \mathrm{Aut}(N)$ a group homomorphism. Then the *semidirect product of N and H with respect to σ* denoted by $N \rtimes_\sigma H$ is the set $N \times H$ with an operation defined as follows $N \rtimes_\sigma H \times N \rtimes_\sigma H \to N \rtimes_\sigma H$, $((n_1, h_1), (n_2, h_2)) \mapsto (n_1\sigma(h_1)(n_2), h_1h_2)$.

For N and H groups and $\sigma : H \to \mathrm{Aut}(N)$ a group homomorphism one can show that $N \rtimes_\sigma H$ is a group. When it is clear what σ is we omit σ in the notation.

Lemma 2.20. *Let G be a group such that $G = N \rtimes H$. Suppose that $K \lhd N$ and $K \lhd G$ then $G/K \cong (N/K) \rtimes H$.*

Proof. The isomorphism is given by $\varphi : (N \rtimes H)/K \to (N/K) \rtimes H$, $(\bar{n}, \bar{h}) \mapsto (\bar{n}, h)$. □

3 Groups of squarefree order

Definition 3.1. A positive integer is *squarefree* if it is not divisible by the square of any prime. A group is *of squarefree order* if it is finite and the order is squarefree.

As a result of the fundamental theorem of abelian groups all finite abelian groups of squarefree order are cyclic.

Proposition 3.2. *Every group of squarefree order is solvable.*

Proof. This theorem is a weaker version of a theorem that can be found here [3, p. 160]. □

The following classical theorem asserts that every group of squarefree order is a semidirect product of two cyclic groups.

Theorem 3.3. *Suppose that G is a finite group. Then the following statements are equivalent:*

1. *G is a group of squarefree order;*
2. *G/G' and G' are cyclic with squarefree and coprime orders.*

Proof. We have that (1) $\implies$ (2) is a weaker version of a theorem that can be found here [3, p. 161]. Furthermore (2) $\implies$ (1) follows directly from Lagrange's theorem. □

As a result of the *Schur-Zassenhaus theorem* [3, p. 82] it follows that every group of squarefree order is a semidirect product of G' and G/G'.

Proposition 3.4. *Let G be of squarefree order. Then* $(\mathrm{Aut}(G))^{(2)} = \{1\}$.

Proof. Let G be a group of squarefree order. By Theorem 3.3 it follows that G/G' and G' are cyclic. Therefore $\mathrm{Aut}(G') \times \mathrm{Aut}(G/G')$ is abelian. Consider

$$\varphi : \mathrm{Aut}(G) \to \mathrm{Aut}(G') \times \mathrm{Aut}(G/G'),\ \sigma \mapsto (\sigma|_{G'}, (gG' \mapsto \sigma(g)G'))$$

Since $\sigma \in \mathrm{Aut}(G)$ it is clear that $gG' \mapsto \sigma(g)G'$ is an automorphism. Furthermore it is clear that φ is a homomorphism. Since $\mathrm{Aut}(G') \times \mathrm{Aut}(G/G')$ is abelian it follows that $\mathrm{Aut}(G)/\ker(\varphi) \cong \varphi(\mathrm{Aut}(G))$ is abelian. Therefore $(\mathrm{Aut}(G)/\ker(\varphi))' = \{1\}$. Now observe that

$$\sigma \in \ker(\varphi) \iff \sigma|_{G'} = \mathrm{id} \text{ and for all } g \in G \text{ it holds that } \sigma(g)G' = gG'.$$

For all $\sigma \in \ker(\varphi)$ we define $a : \ker(\varphi) \times G \to G'$ such that $\sigma(g) = ga(\sigma, g)$. Let the group $(G')^G$ be the product of $\#G$ copies of G'. We consider

$$\psi : \ker \varphi \to (G')^G,\ \sigma \mapsto (a(\sigma, g))_{g \in G}.$$

For $\sigma, \tau \in \ker(\varphi)$ it follows that

$$ga(\tau\sigma, g) = \tau\sigma(g) = \tau(ga(\sigma, g)) = \tau(g)\tau(a(\sigma, g)) = \tau(g)a(\sigma, g) = ga(\tau, g)a(\sigma, g).$$

Hence ψ is a homomorphism. Furthermore note that

$$\sigma \in \ker(\psi) \iff \text{for all } g \in G : a(\sigma, g) = 1 \iff \text{for all } g \in G \text{ it holds that } \sigma(g) = g.$$

Hence $\ker(\psi)$ is trivial thus ψ is injective. Now since $(G')^G$ is abelian it follows that $\ker(\varphi)$ is abelian thus $(\ker(\varphi))' = \{1\}$. We conclude that $(\mathrm{Aut}(G))^{(2)} = \{1\}$. □

4 The socle of a group

Definition 4.1. Let G be a group and H a normal subgroup. Then H is called a *maximal normal subgroup* if $\#\{N \lhd G : H \subset N\} = 2$.

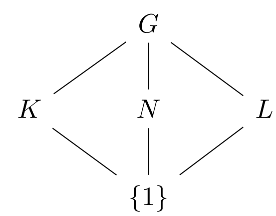
Lemma 4.2. *Let G be a group and N a normal subgroup of G. Then N is a maximal normal subgroup of G if and only if G/N is simple.*

Proof. We leave the proof as an exercise for the reader. □

A consequence of Lemma 4.2 is that every non-trivial group with a composition series has at least one maximal normal subgroup.

Proposition 4.3. *Let G be a group and let K and L be maximal normal subgroups of G such that $K \cap L = \{1\}$ and $K \neq L$. If there exists a proper non-trivial normal subgroup N of G such that $N \neq K$ and $N \neq L$, then G is finite.*

Proof. We have that $K \subset KL$ but K is a maximal normal subgroup hence $KL = G$ or $KL = K$. Since $K \cap L = \{1\}$ the second case does not hold. By Lemma 2.4 it follows that $G \cong K \times L$. Furthermore since K and L are maximal normal subgroups it follows by Lemma 4.2 that K and L are simple. Now suppose that N is a proper non-trivial normal subgroup of G and assume that $N \neq K$ and $N \neq L$. We consider the following diagram:



It follows that the series above are composition series. Hence N is simple. It now follows that $G = NL$ and by Lemma 2.4 it follows that $[K, N] = \{1\}$ and $[K, L] = \{1\}$. We now have that $N \subset \mathrm{C}_G(K)$ and $L \subset \mathrm{C}_G(K)$ hence $G = NL \subset \mathrm{C}_G(K)$. Therefore $G = \mathrm{C}_G(K)$ and K is abelian. Analogously it follows that L is abelian. Since K and L are abelian and simple and $G = KL$ it follows that G is finite. □

A consequence of Theorem 2.17 is that the "length" of a composition series of G does not depend on the composition series but does only depend on G. This gives rise to the following definition.

Definition 4.4. Let G be a group with a composition series

$$\{1\} = G_t \subset G_{t-1} \subset \cdots \subset G_1 \subset G_0 = G.$$

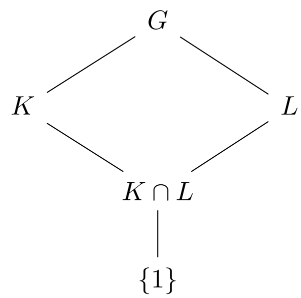
Then, the *composition length* of the composition series is equal to t. Denote the composition length by $\ell(G) = t$.

For any group G that does not have a composition series we use the convention that $\ell(G) = \infty$.

The following proposition now has a surprisingly short proof. The result follows almost directly from the induction hypothesis.

Proposition 4.5. *Let G be a group. Then G has only a finite number of distinct composition series.*

Proof. If G does not have a composition series, then we are immediately done. Suppose that G has a composition series. We will show that G has only finitely many distinct subnormal subgroups. We will proceed by induction on the length of the composition series of G. Suppose that $\ell(G) = 0$. Then $G = \{1\}$. Next assume $\ell(G) > 0$ and for any group H with $\ell(H) < \ell(G)$ assume that H has only finitely many distinct subnormal subgroups. Let K be a maximal normal subgroup of G. Then it follows that $\ell(K) < \ell(G)$ hence by the induction hypothesis K has only finitely many subnormal subgroups. It remains to show that there are only finitely many maximal normal subgroups. We will now look at two cases: suppose that L is a maximal normal subgroup of G such that $K \cap L = \{1\}$. Then it follows by Proposition 4.3 that G is finite or G has precisely 4 normal subgroups. In both cases G has only finitely many maximal normal subgroups. For the second case let L be a maximal normal subgroup of G such that $K \cap L \neq \{1\}$. We consider the following diagram:



By the induction hypothesis it follows that

$$\#\{K \cap L : L \text{ maximal normal subgroup of } G, K \neq L\} < \infty.$$

It now follows that $\#\{G/(K \cap L) : K \cap L \neq \{1\},\ L \text{ maximal normal subgroup of } G, K \neq L\} < \infty$. Now note that $K \cap L$ is normal in G and $\ell(G/(K \cap L)) < \ell(G)$ thus by the induction hypothesis it follows that $\#\{L/(K \cap L) : K \cap L \neq \{1\},\ L \text{ maximal normal subgroup of } G, K \neq L\} < \infty$. By the fourth isomorphism theorem it follows that

$$\{L \text{ maximal normal subgroup of } G : K \cap L \neq \{1\}, K \neq L\} \xrightarrow{\sim} \{L/(K \cap L) : K \cap L \neq 1,\ L \text{ maximal normal subgroup of } G, K \neq L\}$$

is a bijection. We conclude that there are only finitely many maximal normal subgroups of G. $\square$

Definition 4.6. Let G be a group and H a normal subgroup. Then H is called a *minimal normal subgroup* if $\#\{N \lhd G : N \subset H\} = 2$.

From the proof of Proposition 4.5 it follows that a group that admits a composition series has only finitely many normal subgroups. Therefore a non-trivial group that admits a composition series always has at least one minimal normal subgroup.

Definition 4.7. The *socle* of a group G denoted by $\mathrm{Soc}(G)$ is the subgroup

$$\mathrm{Soc}(G) = \langle N : N \text{ is a minimal normal subgroup of } G \rangle.$$

Lemma 4.8. *Let G be a non-trivial group that admits a composition series and let $N \lhd G$ be a normal subgroup and $H \lhd G$ a minimal normal subgroup. Then $H \subset N$ or $HN \cong H \times N$.*

Proof. By minimality of H we have that $H \subset N$ or $H \cap N = \{1\}$. By Lemma 2.4 the second case implies that $HN \cong H \times N$. □

This theorem implies that any two distinct minimal normal subgroups of a group centralize each other.

Theorem 4.9. *The socle of a group that admits a composition series is a direct product of minimal normal subgroups.*

Proof. Let $\mathcal{N} := \{N : N \text{ is a minimal normal subgroup of } G\}$. By Proposition 4.5 it follows that $\mathcal{N}$ is finite. For $\mathcal{S} \subset \mathcal{N}$ and $c_{\mathcal{S}} := \#\mathcal{S}$ we define

$$\varphi_{\mathcal{S}} : \prod_{N \in \mathcal{S}} N \to G,\ (n_1, n_2, \ldots, n_{c_{\mathcal{S}}}) \mapsto n_1 n_2 \cdots n_{c_{\mathcal{S}}}.$$

As a consequence of Lemma 4.8 any two distinct minimal normal subgroups of G centralize each other. Hence for every $\mathcal{S} \subset \mathcal{N}$ the map $\varphi_{\mathcal{S}}$ is a group homomorphism. Note that $\mathrm{im}(\varphi_{\mathcal{N}}) = \mathrm{Soc}(G)$. We now fix $\bar{\mathcal{S}} \subset \mathcal{N}$ as small as possible such that $\mathrm{im}(\varphi_{\bar{\mathcal{S}}}) = \mathrm{Soc}(G)$. It follows for all $1 \leq i \leq c_{\bar{\mathcal{S}}}$ that $N_i \not\subset N_1 N_2 \cdots N_{i-1}$. Hence by Lemma 4.8 for all $1 \leq i \leq c_{\bar{\mathcal{S}}}$ it follows that $N_i \cap (N_1 N_2 \cdots N_{i-1}) = \{1\}$. We conclude that $\varphi_{\bar{\mathcal{S}}}$ is an isomorphism. □

5 Squarefree groups

We sometimes say that a group G that admits a composition series is "built up" from its composition factors. We can now introduce a new notion of "squarefree" in which each composition factor "occurs" once in a composition series of G. We formalize this in the following definition.

Definition 5.1. Let G be a group. Then G is called *squarefree* if it admits a composition series and for a composition series of G the composition factors are pairwise non-isomorphic.

Note that by the equivalence of any two composition series this is independent of the choice of the composition series.

Lemma 5.2. *A group of squarefree order is squarefree.*

Proof. Let G be a group of squarefree order. Then G is finite thus G has a composition series

$$\{1\} = G_t \lhd G_{t-1} \lhd \cdots \lhd G_2 \lhd G_1 \lhd G_0 = G.$$

Now note that

$$\#G = [G : G_1][G_1 : G_2] \cdots [G_{t-2} : G_{t-1}]\#G_{t-1}.$$

Since G is a group of squarefree order we have that $[G_i : G_{i+1}] \neq [G_j : G_{j+1}]$ for all $i, j \in \{0, 1, \ldots, t-1\}$, $i \neq j$. We conclude that the composition factors of G are pairwise non-isomorphic. □

The following theorem states a relation between groups of squarefree order and squarefree groups.

Theorem 5.3. *A group G is of squarefree order if and only if G is squarefree and solvable.*

Proof. " $\Rightarrow$ " This is clear from Proposition 3.2 and Lemma 5.2.

" $\Leftarrow$ " Suppose that G is squarefree and solvable. Then there exists a composition series

$$\{1\} = G_t \lhd G_{t-1} \lhd \ldots \lhd G_1 \lhd G_0 = G.$$

We have that G is solvable thus G_i/G_{i+1} and G_i are solvable for $i \in \{0, 1, \ldots, t-1\}$. But by assumption we also have that G_i/G_{i+1} is simple hence $(G_i/G_{i+1})' = \{1\}$. Therefore G_i/G_{i+1} is abelian for all $i \in \{0, 1, \ldots, t-1\}$. Hence every composition factor of G is cyclic of prime order. Furthermore we note that G is squarefree hence all composition factors are pairwise non-isomorphic. We conclude that G is of squarefree order. □

For a group G and a normal subgroup N it is clear that the following equivalence holds: G is of squarefree order if and only if N and G/N have squarefree and coprime orders. Then the question arises if we can say something similar for squarefree groups. The following lemma gives an answer to this question.

Proposition 5.4. *Let G be a squarefree group and $N \lhd G$ a normal subgroup. Then N and G/N are both squarefree groups.*

Proof. We have by Lemma 2.18 that N is part of a composition series of G. We can now consider the composition series

$$\{1\} = G_t \subset G_{t-1} \subset \cdots G_{k+1} \subset N \subset G_{k-1} \subset \cdots \subset G_1 \subset G_0 = G$$

for $k \in \{0, 1, \ldots, t-1\}$. Since G is squarefree

$$\{1\} = G_t \subset G_{t-1} \subset \cdots \subset G_{k+1} \subset N$$

is a composition series of N and the composition factors are pairwise non-isomorphic. Therefore N is squarefree. Furthermore consider the normal series

$$\{1\} \subset G_{k-1}/N \subset \cdots \subset G_1/N \subset G_0/N = G/N$$

We have that

$$\frac{G_i/N}{G_{i+1}/N} \cong G_i/G_{i+1}$$

is simple. Since G is squarefree all composition factors of G/N are pairwise non-isomorphic. We conclude that G/N is squarefree. □

To show that the other implication does not hold we will give a counterexample. For p a prime consider $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Take $N = \mathbb{Z}/p\mathbb{Z}$, then G is not squarefree but N and G/N are squarefree.

Another counterexample is the following: let $G = \mathrm{D}_6$ be the dihedral group of order 12 and let $N \subset G$ be the cyclic group of order 6. Then N and G/N are squarefree but G is not squarefree.

Lemma 5.5. *Let G be a squarefree group and $N \lhd G$ a normal subgroup. If G/N is simple, then N is a characteristic subgroup of G.*

Proof. Let $\varphi \in \mathrm{Aut}(G)$ and we define $M := \varphi(N)$. Assume that $M \neq N$. We note that M is a normal subgroup of G. We will show $G/M \cong G/N$. For $\mathrm{can} : G \to G/N$ the canonical map consider the following:

$$G \xrightarrow{\varphi^{-1}} G \xrightarrow{\mathrm{can}} G/N.$$

We observe that $\mathrm{can} \circ \varphi^{-1}$ is surjective and $\ker(\mathrm{can} \circ \varphi^{-1}) = M$. The first isomorphism theorem now gives $G/M \cong G/N$. Since $M \neq N$ and G/N is simple it follows by Theorem 2.12 that $G/(N \cap M) \cong G/N \times G/M$. Since $M \cap N$ is normal in G and G is squarefree it follows by Proposition 5.4 that $G/(N \cap M)$ is squarefree. Now observe that

$$\{1\} \lhd G/N \times \{1\} \lhd G/N \times G/\varphi(N)$$

is a composition series of $G/N \times G/\varphi(N)$. Since $G/N \cong G/\varphi(N)$ it follows that $G/N \times G/\varphi(N)$ is not squarefree. This is a contradiction. □

Lemma 5.6. *Let G be a squarefree group. Then $G^{(2)} = G^{(3)}$.*

Proof. Let G be a squarefree group. We have by Proposition 5.4 that $G/G^{(3)}$ is squarefree. Furthermore note that $\left(G/G^{(3)}\right)^{(3)} = G^{(3)}/G^{(3)} = \{1\}$. Therefore $G/G^{(3)}$ is solvable. By Theorem 5.3 it follows that $G/G^{(3)}$ is of squarefree order. Therefore by Theorem 3.3 it follows that $(G/G^{(3)})' = G'/G^{(3)}$ is cyclic hence $G^{(2)}/G^{(3)} = \{1\}$. We conclude that $G^{(2)} = G^{(3)}$. □

Theorem 5.7. *Let G be a squarefree group and let H be a subgroup of G. Then the following are equivalent:*

1. *the subgroup H of G is subnormal in G;*
2. *the subgroup H of G is normal in G;*
3. *the subgroup H of G is characteristic in G.*

Proof. The implications (3) $\implies$ (2) and (2) $\implies$ (1) follow directly. We will show that (1) $\implies$ (3). We may assume that H is a proper subgroup of G. Suppose that H is subnormal in G then by Lemma 2.18 it follows that H is part of a composition series of G. Since the property of being characteristic is transitive it follows by induction on the length of a composition series of G containing H and using Lemma 5.5 that H is characteristic in G. □

Lemma 5.8. *Let G be a squarefree group and let N be a normal solvable subgroup of G''. Then $N \subset \mathrm{Z}(G'')$.*

Proof. We have by Lemma 5.6 that $G^{(2)} = G^{(3)}$ so $G^{(2)}$ is perfect. By Theorem 5.7 and Proposition 5.4 we have that N is squarefree. Furthermore since N is solvable we have by Theorem 5.3 that N is of squarefree order. Therefore by Proposition 3.4 it follows that $\mathrm{Aut}(N)$ is solvable. Hence by Theorem 2.7 it follows for any homomorphism $\varphi : G'' \to \mathrm{Aut}(N)$ that φ is trivial. Therefore G'' acts trivially on N i.e. for all $g \in G''$ and $n \in N$ it follows that $gng^{-1} = n$. We conclude that $N \subset \mathrm{Z}(G'')$. □

Lemma 5.9. *Let G be squarefree. Then $\mathrm{Z}(G'')$ is cyclic and of squarefree order.*

Proof. We first observe that $\mathrm{Z}(G'') \lhd G'' \lhd G$ hence by Proposition 5.4 we have $\mathrm{Z}(G'')$ is squarefree. Furthermore since $\mathrm{Z}(G'')$ is abelian it follows that $\mathrm{Z}(G'')$ is solvable. Therefore by Theorem 5.3 we have that $\mathrm{Z}(G'')$ is of squarefree order. Now since $\mathrm{Z}(G'')$ is abelian and of squarefree order it follows that $\mathrm{Z}(G'')$ is cyclic. □

Lemma 5.10. *Let G be a squarefree group. If N is a minimal normal subgroup of G then N is simple.*

Proof. Suppose that K is a normal subgroup of N. By Theorem 5.7 we have that K is a normal subgroup of G. Since N is minimal it follows that $K = N$ or $K = \{1\}$. We conclude that N is simple. □

Lemma 5.11. *The socle of a squarefree group is a direct product of finitely many simple groups.*

Proof. This follows immediately from Theorem 4.9 and Lemma 5.10. □

Lemma 5.12. *Let G be a squarefree group. Then every minimal normal subgroup of $G''/\mathrm{Z}(G'')$ is non-abelian and simple.*

Proof. We have by Proposition 5.4 that $G''/\mathrm{Z}(G'')$ is squarefree. Suppose that $M/\mathrm{Z}(G'')$ is a minimal normal subgroup of $G''/\mathrm{Z}(G'')$. Then it follows that $M/\mathrm{Z}(G'')$ is simple by Lemma 5.10. We consider the following diagram:

$$\begin{array}{ccccc}
\{1\} & \rightarrowtail & M/\mathrm{Z}(G'') & \rightarrowtail & G''/\mathrm{Z}(G'') \\
\uparrow\!\!\uparrow & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\
\mathrm{Z}(G'') & \rightarrowtail & M & \rightarrowtail & G''
\end{array}$$

Suppose that $M/\mathrm{Z}(G'')$ is abelian, then $M/\mathrm{Z}(G'')$ is solvable. Furthermore we note that $\mathrm{Z}(G'')$ is solvable hence M is solvable. It now follows by Lemma 5.8 that $M \subset \mathrm{Z}(G'')$. Therefore $M = \mathrm{Z}(G'')$. This is a contradiction since $M/\mathrm{Z}(G'')$ is a minimal normal subgroup. □

To increase the clarity we introduce the notation $\Gamma(G) := G''/\mathrm{Z}(G'')$.

Lemma 5.13. *If G is a squarefree group, then $\mathrm{Z}(\mathrm{Soc}(\Gamma(G))) = \{1\}$.*

Proof. It follows from Theorem 4.9 that $\mathrm{Soc}(\Gamma(G)) = S_1 \times S_2 \times \cdots \times S_t$ for S_i a minimal normal subgroup of $\Gamma(G)$, $i \in \{1, 2, \ldots, t\}$. Therefore

$$\mathrm{Z}(\mathrm{Soc}(\Gamma(G))) \cong \mathrm{Z}(S_1 \times S_2 \times \cdots \times S_t) = \mathrm{Z}(S_1) \times \mathrm{Z}(S_2) \times \cdots \times \mathrm{Z}(S_t).$$

Now observe that from Lemma 5.12 it follows that S_i is non-abelian and simple for all $i \in \{1, 2, \ldots, t\}$. Hence $\mathrm{Z}(S_i) = \{1\}$. We conclude that $\mathrm{Z}(\mathrm{Soc}(\Gamma(G))) = \{1\}$. □

Lemma 5.14. *Let G be a group and $H \subset G$ a subgroup. Then $\mathrm{Z}(H) = \mathrm{C}_G(H) \cap H$.*

Proof. We leave the proof as an exercise for the reader. □

Proposition 5.15. *Let G be a squarefree group and N a normal subgroup of $\Gamma(G)$ such that $N \cap \mathrm{Soc}(\Gamma(G)) = \{1\}$. Then $N = \{1\}$.*

Proof. Assume that $N \neq \{1\}$. Since N is a normal subgroup of $\Gamma(G)$ there exists a composition series

$$\{1\} = G_t \subset G_{t-1} \subset \cdots \subset G_1 \subset G_0 = \Gamma(G)$$

such that $G_i = N$ for some $i \in \{0, 1, \ldots, t-1\}$. By Theorem 5.7 it follows that G_{t-1} is a normal subgroup of G. Since G is squarefree it follows that G_{t-1} is a minimal normal subgroup of $\Gamma(G)$ and $G_{t-1} \subset G_i = N$. Furthermore by definition of the socle it holds that $G_{t-1} \subset \mathrm{Soc}(\Gamma(G))$. Hence $G_{t-1} \subset N \cap \mathrm{Soc}(\Gamma(G))$ and since G_{t-1} is non-trivial it follows that $N \cap \mathrm{Soc}(\Gamma(G)) \neq \{1\}$. □

6 Schreier groups

In this chapter we will consider squarefree groups with some extra property.

Definition 6.1. A group G is a *Schreier group* if G has a composition series and for every composition factor S of G the group $\mathrm{Out}(S)$ is solvable.

Theorem 6.2 (Schreier's Conjecture)**.** *Let G be a finite simple group. Then* $\mathrm{Out}(G)$ *is solvable.*

The only proof known of this theorem is computing the outer automorphism group of every finite simple group [1]. All finite simple groups are described in *the classification of finite simple groups* [2].

Proposition 6.3. *Every finite group is a Schreier group.*

Proof. Every finite group has a composition series. Furthermore every composition factor of G is simple. Hence by Theorem 6.2 it follows that the outer automorphism group of every composition factor of G is solvable. □

Again we introduce the notation $\Gamma(G) := G''/Z(G'')$.

Proposition 6.4. *Let G be squarefree Schreier group. Then* $\mathrm{Soc}(\Gamma(G)) = \Gamma(G)$.

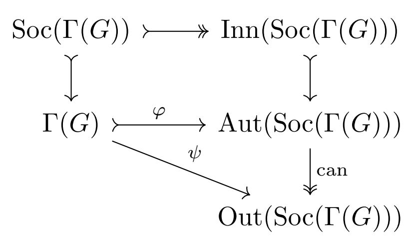
Proof. The map

$$\varphi : \Gamma(G) \to \mathrm{Aut}(\mathrm{Soc}(\Gamma(G))), \quad g \mapsto (x \mapsto gxg^{-1})$$

is a group homomorphism and we have that $\ker(\varphi) = \mathrm{C}_{\Gamma(G)}(\mathrm{Soc}(\Gamma(G)))$. By Lemma 5.14 it follows that

$$\mathrm{Z}(\mathrm{Soc}(\Gamma(G))) = \mathrm{C}_{\Gamma(G)}(\mathrm{Soc}(\Gamma(G))) \cap \mathrm{Soc}(\Gamma(G)).$$

Furthermore by Lemma 5.12 it follows that $\mathrm{Z}(\mathrm{Soc}(\Gamma(G))) = \{1\}$ hence $\mathrm{Soc}(\Gamma(G)) \cong \mathrm{Inn}(\mathrm{Soc}(\Gamma(G)))$. It now follows by Proposition 5.15 that $\ker(\varphi) = \{1\}$ hence φ is injective. For $\mathrm{can} : \mathrm{Aut}(\mathrm{Soc}(\Gamma(G))) \to \mathrm{Out}(\mathrm{Soc}(\Gamma(G)))$ the canonical map we define $\psi := \mathrm{can} \circ \varphi$. Consider the following commutative diagram:



It follows that $\mathrm{Soc}(\Gamma(G)) \subset \ker(\psi)$. Now let $\gamma \in \ker(\psi)$ then $\varphi(\gamma) \in \mathrm{Inn}(\mathrm{Soc}(\Gamma(G)))$. Furthermore there exists $\delta \in \mathrm{Soc}(\Gamma(G))$ such that $\varphi(\delta) = \varphi(\gamma)$. Now since φ is injective it follows that $\gamma = \delta$. Hence we have $\ker(\psi) = \mathrm{Soc}(\Gamma(G))$. We will now show that ψ is trivial. We observe that from Theorem 4.9 and Lemma 5.12 it follows that

$$\mathrm{Soc}(\Gamma(G)) \cong S_1 \times \cdots \times S_t$$

for S_i non-abelian and simple and $S_i \ncong S_j$ for $i, j \in \{1, 2, \ldots, t\},\ i \neq j$. Furthermore by Proposition 5.4 and Theorem 5.7 it follows that $\mathrm{Soc}(\Gamma(G))$ is a squarefree group. Therefore by Theorem 5.7 it follows for all $i \in \{1, 2, \ldots, t\}$ that S_i is a characteristic subgroup of $\prod_{i=1}^{t} S_i$. Hence

$$\mathrm{Aut}\left(\prod_{i=1}^{t} S_i\right) = \prod_{i=1}^{t} \mathrm{Aut}(S_i)$$

and thus $\mathrm{Out}\left(\prod_{i=1}^{t} S_i\right) = \prod_{i=1}^{t} \mathrm{Out}(S_i)$. Since G is a Schreier group we have that $\mathrm{Out}(\mathrm{Soc}(\Gamma(G))$ is solvable. Furthermore by Lemma 5.6 it follows that $\Gamma(G) = (\Gamma(G))'$. Hence by Theorem 2.7 it follows that every homomorphism $\psi : \Gamma(G) \to \mathrm{Soc}(\mathrm{Out}(\Gamma(G)))$ is trivial. Hence $\ker(\psi) = \Gamma(G)$. We conclude that $\Gamma(G) = \mathrm{Soc}(\Gamma(G))$. □

Lemma 6.5. *Let G be a squarefree Schreier group. Then $\Gamma(G)$ is a direct product of finitely many pairwise non-isomorphic non-abelian simple minimal normal subgroups.*

Proof. From Proposition 6.4 it follows that $\mathrm{Soc}(\Gamma(G)) = \Gamma(G)$. Now by Theorem 4.9 it follows that $\Gamma(G)$ is a direct product of minimal normal subgroups. By Lemma 5.12 it follows that every minimal normal subgroup is non-abelian and simple. □

Theorem 6.6. *Let G be a Schreier group. Then the following are equivalent:*

1. *the group G is squarefree;*
2. *the quotient groups G/G', G'/G'' and $\mathrm{Z}(G'')$ are cyclic with squarefree and pairwise coprime orders and in addition $G''/\mathrm{Z}(G'')$ is the direct product of finitely many pairwise non-isomorphic non-abelian simple groups.*

Proof. We will prove both implications.

"$1 \Rightarrow 2$" Assume that G is squarefree. Since G/G' and G'/G'' are abelian it follows that G/G' and G'/G'' are solvable. Hence by Theorem 5.3 it follows that G/G' and G'/G'' are of squarefree orders. Since both quotient groups are abelian and of squarefree orders it follows that both quotient groups are cyclic. By Lemma 5.9 it follows that $\mathrm{Z}(G'')$ is cyclic and of squarefree order. Note that the composition factors of G/G', G'/G'' and $\mathrm{Z}(G'')$ are cyclic of prime order. Since G is squarefree it follows that $[G : G'], [G' : G'']$ and $\#\mathrm{Z}(G'')$ are pairwise coprime.

Furthermore by Lemma 6.5 it follows that $G''/\mathrm{Z}(G'')$ is the direct product of finitely many pairwise non-isomorphic non-abelian simple groups.

"$1 \Leftarrow 2$" Consider the following

$$\{1\} \lhd \mathrm{Z}(G'') \lhd G'' \lhd G' \lhd G.$$

Since G/G', G'/G'' and $\mathrm{Z}(G'')$ are squarefree and have coprime orders it follows that G/G', G'/G'' and $\mathrm{Z}(G'')$ have composition series and the composition factors of those series are cyclic of prime order and are pairwise non-isomorphic. By assumption we can write $G''/\mathrm{Z}(G'') = \prod_{i=1}^{t} S_i$, where S_i is non-abelian and simple and $S_i \not\cong S_j$ for $0 < i, j \leq t$, $i \neq j$. We can now construct a normal series

$$\{1\} \lhd S_1 \lhd S_1 \times S_2 \lhd \cdots \lhd \prod_{i=1}^{t-1} S_i \lhd \prod_{i=1}^{t} S_i = G''/\mathrm{Z}(G'')$$

with factor groups $\prod_{i=1}^{k} S_i \Big/ \prod_{i=1}^{k-1} S_i \cong S_k$ for $0 < k \leq t$. It follows that the factor groups of $G''/\mathrm{Z}(G'')$ are simple and pairwise non-isomorphic. Hence the normal series above is a composition series. Since the composition factors are non-abelian and simple it follows that these composition factors are pairwise non-isomorphic with the composition factors of G/G', G'/G'' and $\mathrm{Z}(G'')$. It now follows that G is squarefree. □

7 Examples

7.1 Finite squarefree groups

The reader may wonder if there exists a squarefree groups such that every factor group described in Theorem 6.6 is non-trivial. The answer to this question is yes. We will show how to construct such groups and we will give an example of a group with minimal order in this class of examples.

We assume the reader is familiar with linear groups over finite fields. Let $q = p^d$ a prime power. We denote $\Gamma\mathrm{L}(n, \mathbb{F}_q) := \mathrm{GL}(n, \mathbb{F}_q) \rtimes \langle \mathrm{Frob} \rangle$ where $\langle \mathrm{Frob} \rangle$ is the Galois group of $\mathbb{F}_q$ over $\mathbb{F}_p$, which acts entry-wise on $\mathrm{GL}(n, \mathbb{F}_q)$.

Definition 7.1. Let G be an abelian group and let $n \in \mathbb{Z}$. Then the *n-torsion group* is the group denoted by $G[n] = \{g \in G : g^n = 1\}$.

We note that for every $n \in \mathbb{Z}$ it follows that $G[n]$ is a characteristic subgroup of G.

Theorem 7.2. *Let $n > 1$ and $q = p^d$ for a prime p and $d \in \mathbb{Z}_{>0}$. Let $a, b, c \in \mathbb{Z}_{>0}$ and assume that $a \mid d$, $b \mid q-1$ and $c \mid \gcd(n, q-1)$. We now define*

$$\Lambda(b, n, q) := \{x \in \mathrm{GL}(n, \mathbb{F}_q) : \det x \in \mathbb{F}_q^*[b]\}.$$

Then for $\kappa(c, n, q) := \gcd(n, q-1)/c$ it follows that $G := \Lambda(b, n, q)/\mathbb{F}_q^*[\kappa(c, n, q)] \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle$ is a group and we derive the following properties:

1. *for $q > 3$ and $\gcd(p^{\frac{d}{a}} - 1, b) = 1$ it follows that $G' = \Lambda(b, n, q)/\mathbb{F}_q^*[\kappa(c, n, q)]$;*
2. *for $q > 3$ it follows that $(\Lambda(b, n, q)/\mathbb{F}_q^*[\kappa(c, n, q)])' = \mathrm{SL}(n, \mathbb{F}_q)/\mathbb{F}_q^*[\kappa(c, n, q)]$;*
3. *for $q > 3$ it follows that $\mathrm{Z}(\mathrm{SL}(n, \mathbb{F}_q)/\mathbb{F}_q^*[\kappa(c, n, q)]) = \mathbb{F}_q^*[n]/\mathbb{F}_q^*[\kappa(c, n, q)]$;*
4. *for $q > 3$, $\gcd(p^{\frac{d}{a}} - 1, b) = 1$ and a, b and c squarefree and pairwise coprime it follows that G is a squarefree group and $\#G = \frac{abc}{\gcd(n,q-1)} q^{\frac{n(n-1)}{2}} \prod_{i=2}^{n} (q^i - 1)$.*

The following diagram clarifies what Theorem 7.2 states. All the arrows are the obvious ones.

$$\begin{array}{ccccccccc}
\{1\} & \rightarrowtail & \mathbb{F}_q^*[n] & \rightarrowtail & \mathrm{SL}(n,\mathbb{F}_q) & \rightarrowtail & \mathrm{GL}(n,\mathbb{F}_q) & \rightarrowtail & \Gamma\mathrm{L}(n,\mathbb{F}_q) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\{1\} & \rightarrowtail & \mathbb{F}_q^*[n] & \rightarrowtail & \mathrm{SL}(n,\mathbb{F}_q) & \rightarrowtail & \Lambda(b,n,q) & \rightarrowtail & \Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{1\} & \rightarrowtail & \frac{\mathbb{F}_q^*[n]}{\mathbb{F}_q^*[\kappa(c,n,q)]} & \rightarrowtail & \frac{\mathrm{SL}(n,\mathbb{F}_q)}{\mathbb{F}_q^*[\kappa(c,n,q)]} & \rightarrowtail & \frac{\Lambda(b,n,q)}{\mathbb{F}_q^*[\kappa(c,n,q)]} & \rightarrowtail & \frac{\Lambda(b,n,q)}{\mathbb{F}_q^*[\kappa(c,n,q)]} \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle
\end{array}$$

To prove Theorem 7.2 we will first consider the middle row of the diagram.

Lemma 7.3. *For $q > 3$ and $\gcd(p^{\frac{d}{a}} - 1, b) = 1$ it follows that $(\Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle)' = \Lambda(b,n,q)$.*

Proof. " $\subset$ " By Lemma 2.20 it follows that

$$\frac{\Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle}{\Lambda(b,n,q)} \cong \langle \mathrm{Frob}^{\frac{d}{a}} \rangle.$$

Hence $(\Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle)' \subset \Lambda(b,n,q)$.
" $\supset$ " Since $q > 3$ it follows that $\mathrm{SL}(n,\mathbb{F}_q)' = \mathrm{SL}(n,\mathbb{F}_q)$, this can be found here [5, p. 45-46]. It now follows that $\mathrm{SL}(n,\mathbb{F}_q) \subset \Lambda(b,n,q)' \subset (\Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle)'$. We now note that

$$\left(\frac{\Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle}{\mathrm{SL}(n,\mathbb{F}_q)}\right)' = \frac{\mathrm{SL}(n,\mathbb{F}_q)(\Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle)'}{\mathrm{SL}(n,\mathbb{F}_q)} = \frac{(\Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle)'}{\mathrm{SL}(n,\mathbb{F}_q)}.$$

We have by Lemma 2.20 and the first isomorphism theorem that

$$\frac{\Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle}{\mathrm{SL}(n,\mathbb{F}_q)} \cong \frac{\Lambda(b,n,q)}{\mathrm{SL}(n,\mathbb{F}_q)} \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle \cong \mathbb{F}_q^*[b] \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle.$$

Let $\zeta \in \mathbb{F}_q^*[b]$ be a generator. In $\mathbb{F}_q^*[b] \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle$ we have that

$$[\mathrm{Frob}^{\frac{d}{a}}, \zeta] = \mathrm{Frob}^{\frac{d}{a}}(\zeta)\zeta^{-1} = \zeta^{p^{\frac{d}{a}}-1}.$$

Now since $\gcd(p^{\frac{d}{a}} - 1, b) = 1$ it follows that $\zeta^{p^{\frac{d}{a}}-1}$ has order b thus $\langle \zeta^{p^{\frac{d}{a}}-1} \rangle = \mathbb{F}_q^*[b]$. It now follows that $\mathbb{F}_q^*[b] \subset (\mathbb{F}_q^*[b] \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle)'$. Now note that

$$(\mathbb{F}_q^*[b] \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle)' \cong \left(\frac{\Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle}{\mathrm{SL}(n,\mathbb{F}_q)}\right)' \cong \frac{(\Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle)'}{\mathrm{SL}(n,\mathbb{F}_q)}.$$

Since $\mathbb{F}_q^*[b] \cong \Lambda(b,n,q)/\mathrm{SL}(n,\mathbb{F}_q)$ it follows that $\Lambda(b,n,q) \subset (\Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle)'$. □

Lemma 7.4. *For $q > 3$ it follows that $\Lambda(b,n,q)' = \mathrm{SL}(n,\mathbb{F}_q)$.*

Proof. By the first isomorphism theorem it follows that $\Lambda(b,n,q)/\mathrm{SL}(n,\mathbb{F}_q) \cong \mathbb{F}_q^*[b]$. Hence

$$\left(\frac{\Lambda(b,n,q)}{\mathrm{SL}(n,\mathbb{F}_q)}\right)' = \frac{\Lambda(b,n,q)'\mathrm{SL}(n,\mathbb{F}_q)}{\mathrm{SL}(n,\mathbb{F}_q)} = \{1\}.$$

Thus we have $\Lambda(b,n,q)' \subset \mathrm{SL}(n,\mathbb{F}_q)$. The other inclusion follows from the proof of Lemma 7.3. We conclude that $\Lambda(b,n,q)' = \mathrm{SL}(n,\mathbb{F}_q)$. □

We can now prove the main theorem of this chapter.

Proof of Theorem 7.2. We will first show that G is indeed a group. We note that $\mathrm{Z}(\mathrm{GL}(n,\mathbb{F}_q)) = \mathbb{F}_q^*$ hence we have the following:

$$\mathbb{F}_q^*[\kappa(c,n,q)] \underset{\text{char}}{\lhd} \mathbb{F}_q^* \underset{\text{char}}{\lhd} \mathrm{GL}(n,\mathbb{F}_q) \lhd \Gamma\mathrm{L}(n,\mathbb{F}_q).$$

It now follows that $\mathbb{F}_q^*[\kappa(c,n,q)]$ is a normal subgroup of $\Gamma\mathrm{L}(n,\mathbb{F}_q)$. Hence $\mathbb{F}_q^*[\kappa(c,n,q)]$ is a normal subgroup of $\Lambda(b,n,q)$. By Lemma 2.20 we now have that

$$(\Lambda(b,n,q) \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle)/\mathbb{F}_q^*[\kappa(c,n,q)] \cong \Lambda(b,n,q)/\mathbb{F}_q^*[\kappa(c,n,q)] \rtimes \langle \mathrm{Frob}^{\frac{d}{a}} \rangle$$

is a group. We note that (1) and (2) follow from Lemma 7.3 and Lemma 7.4. For (3) we have by [5, p. 45-46] that $\mathrm{PSL}(n,\mathbb{F}_q)$ is non-abelian and simple for $q > 3$. It follows that

$$\begin{aligned}\frac{\mathrm{Z}\left(\mathrm{SL}(n,\mathbb{F}_q)/\mathbb{F}_q^*[\kappa(c,n,q)]\right)}{\mathbb{F}_q^*[n]/\mathbb{F}_q^*[\kappa(c,n,q)]} &\subset \mathrm{Z}\left(\frac{\mathrm{SL}(n,\mathbb{F}_q)/\mathbb{F}_q^*[\kappa(c,n,q)]}{\mathbb{F}_q^*[n]/\mathbb{F}_q^*[\kappa(c,n,q)]}\right) \\ &\cong \mathrm{Z}(\mathrm{SL}(n,\mathbb{F}_q)/\mathbb{F}_q^*[n]) = \mathrm{Z}(\mathrm{PSL}(n,\mathbb{F}_q)) = \{1\}.\end{aligned}$$

For (4) we will first determine the order of G. We note that from the proof of Lemma 7.3 it follows that $G/G' \cong \langle \mathrm{Frob}^{\frac{d}{a}} \rangle$ hence $[G : G'] = a$. Furthermore from the proof of Lemma 7.4 it follows that $G'/G'' \cong \mathbb{F}_q^*[b]$. Now since $b \mid q-1$ it follows that $[G', G''] = b$. Furthermore we have that $\#\mathrm{Z}(G'') = c$. By [5, p. 44] it follows that

$$[G'' : \mathrm{Z}(G'')] = \#\mathrm{PSL}(n,\mathbb{F}_q) = \frac{1}{\gcd(n,q-1)} q^{\frac{n(n-1)}{2}} \prod_{i=2}^{n}(q^i - 1).$$

The result now follows by Lagrange's Theorem. Now since we assumed that a, b and c are squarefree and pairwise coprime and $\mathrm{PSL}(n,\mathbb{F}_q)$ is simple for $q > 3$ or $n > 2$ it follows by Theorem 6.6 that G is squarefree. □

We will now give the smallest numerical example of the class of groups described in Theorem 7.2.

Example 7.5. Take $p = d = 3$, $n = 2$, $a = 3$, $b = 13$ and $c = 2$. Then p, d, n, a, b and c satisfy all requirements of Theorem 7.2. We have that $\kappa(2,2,27) = 1$ and $G = \Lambda(13,2,27) \rtimes \langle \mathrm{Frob} \rangle$. Consider the following diagram:

$$\{1\} \underset{2}{\lhd} \mathbb{F}_{27}^*[2] \overset{|\mathrm{PSL}(2,\mathbb{F}_{27})|}{\lhd} \mathrm{SL}(2,\mathbb{F}_{27}) \underset{13}{\lhd} \Lambda(13,2,27) \overset{3}{\lhd} \Lambda(13,2,27) \rtimes \langle \mathrm{Frob} \rangle.$$

It follows that $\#G = 3 \cdot 13 \cdot 2 \cdot 9828 = 766584$.

We leave it as an exercise for the reader to check that this is indeed the smallest example in the class of groups described in Theorem 7.2.

7.2 Squarefree groups of infinite order

We know that the Schreier condition is sufficient for Theorem 6.6. The reader may wonder if the Schreier condition may be omitted from Theorem 6.6. As a result of Schreier's Conjecture we saw that every finite group satisfies the Schreier condition. So when looking for an example of a squarefree group that does not satisfy the Schreier condition we are only interested in groups of infinite order. We will show how to construct such example.

Proposition 7.6. *Let* $\varphi : \mathrm{Aut}(k) \to \mathrm{Aut}(\mathrm{PSL}(2,k))$ *be a group homomorphism given by*

$$\sigma \mapsto \left(\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \pm\left(\begin{smallmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{smallmatrix}\right)\right),$$

and let $\mathrm{can} : \mathrm{Aut}(\mathrm{PSL}(2,k)) \to \mathrm{Out}(\mathrm{PSL}(2,k))$ *be the canonical map. Consider the following commutative diagram:*

$$\begin{array}{ccc} & & \mathrm{Out}(\mathrm{PSL}(2,k)) \\ & \overset{\psi}{\nearrow} & \uparrow\!\!\uparrow \mathrm{can} \\ \mathrm{Aut}(k) & \xrightarrow{\varphi} & \mathrm{Aut}(\mathrm{PSL}(2,k)) \end{array}$$

It follows that $\psi := \mathrm{can} \circ \varphi$ *is injective.*

Proof. Let $\sigma \in \ker(\psi)$. Then it follows that $\varphi(\sigma) \in \mathrm{Inn}(\mathrm{PSL}(2,k))$. Hence there exists some $y \in \mathrm{SL}(2,k)$ such that for all $x \in \mathrm{SL}(2,k)$ one has $\varphi(\sigma)(\pm x) = \pm yxy^{-1}$. Let $x = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$\varphi(\sigma)(\pm x) = \pm\left(\begin{smallmatrix} 1 & \sigma(a) \\ 0 & 1 \end{smallmatrix}\right) = \pm y\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)y^{-1}.$$

Now consider two cases. If $\mathrm{char}(k) = 2$ then $\left(\begin{smallmatrix} 1 & \sigma(a) \\ 0 & 1 \end{smallmatrix}\right) = y\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)y^{-1}$. If $\mathrm{char}(k) \neq 2$ we square both sides. We have

$$[\varphi(\sigma)(\pm x)]^2 = \left(\begin{smallmatrix} 1 & \sigma(2a) \\ 0 & 1 \end{smallmatrix}\right) \text{ and } \left[\pm y\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)y^{-1}\right]^2 = y\left(\begin{smallmatrix} 1 & 2a \\ 0 & 1 \end{smallmatrix}\right)y^{-1}.$$

We conclude that $\varphi(\sigma)(x) = yxy^{-1}$. Let $y = \left(\begin{smallmatrix} q & r \\ s & t \end{smallmatrix}\right)$. Then

$$\varphi(\sigma)(x)y = \left(\begin{smallmatrix} 1 & \sigma(a) \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} q & r \\ s & t \end{smallmatrix}\right) = \pm\left(\begin{smallmatrix} q+\sigma(a)s & r+\sigma(a)t \\ s & t \end{smallmatrix}\right) \text{ and } yx = \left(\begin{smallmatrix} q & r \\ s & t \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} q & qa+r \\ s & sa+t \end{smallmatrix}\right).$$

It now follows that

$$\left(\begin{smallmatrix} q+\sigma(a)s & r+\sigma(a)t \\ s & t \end{smallmatrix}\right) = \left(\begin{smallmatrix} q & qa+r \\ s & sa+t \end{smallmatrix}\right).$$

The equality must hold for all $a \in k$ so if $a = 1$, then it follows that $s = 0$. Furthermore since $\left(\begin{smallmatrix} q & qa+r \\ 0 & t \end{smallmatrix}\right) \in \mathrm{SL}(2,k)$ we have $t = q^{-1}$. We now have the following equality: $\left(\begin{smallmatrix} q & r+\sigma(a)q^{-1} \\ 0 & q^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} q & r+qa \\ 0 & q^{-1} \end{smallmatrix}\right)$. It now follows that $\sigma(a)q^{-1} = aq$ and thus $\sigma(a) = aq^2$. Since σ is a field automorphism it follows that $q = \pm 1$ and $\sigma = \mathrm{id}$. □

Lemma 7.7. *Let* S *be a group and let* I *be an index set such that* $|S| \leq |I|$. *Let* $k = \mathbb{Q}(X_i : i \in I)$. *Then there exists an injective group homomorphism* $\phi : S \to \mathrm{Aut}(k)$.

Proof. By Cayley's Theorem it follows that the map $S \to \mathrm{Sym}(S), (s \mapsto (x \mapsto sx))$ is an injective group homomorphism. Furthermore since $|S| \leq |I|$ there exists an injection $\mathrm{Sym}(S) \to \mathrm{Sym}(I)$. Note that the the map $\mathrm{Sym}(I) \to \mathrm{Aut}(k)$, $\sigma \mapsto (X_i \mapsto X_{\sigma(i)})$ is an injective group homomorphism. Now let $\phi : S \to \mathrm{Aut}(k)$ be the composed map $S \to \mathrm{Sym}(S) \to \mathrm{Sym}(I) \to \mathrm{Aut}(k)$. Then ϕ is the required injective group homomorphism. □

Proposition 7.8. *Let $\rho : H \to \mathrm{Aut}(N)$ be a group homomorphism and let* $\mathrm{can} : \mathrm{Aut}(N) \to \mathrm{Out}(N)$ *be the canonical map. Consider the following diagram:*

$$H \xrightarrow{\ \rho\ } \mathrm{Aut}(N) \xrightarrow{\ \mathrm{can}\ } \mathrm{Out}(N)$$

Then $\mathrm{C}_{N\rtimes_\rho H}(N) = \{1\}$ *if and only if* $\mathrm{Z}(N) = \{1\}$ *and* $\mathrm{can} \circ \rho$ *is injective.*

Proof. Suppose that $(x, y) \in N \rtimes_\rho H$. Then for all $a \in N$ it follows that

$$\begin{aligned}
&(x, y)(a, 1)(x, y)^{-1} = (a, 1) \\
&\iff (1, y)(a, 1)(1, y)^{-1} = (x^{-1}, 1)(a, 1)(x, 1) \\
&\iff (\rho(y)(a), 1) = (x^{-1}ax, 1) \\
&\iff \rho(y)(a) = x^{-1}ax
\end{aligned}$$

" $\Rightarrow$ " Suppose that $\mathrm{C}_{N\rtimes_\rho H}(N) = \{1\}$. Note that $\mathrm{Z}(N) \subset \mathrm{C}_{N\rtimes_\rho H}(N)$ is a subgroup. Hence $\mathrm{Z}(N) = \{1\}$. Furthermore for $y \in \ker(\mathrm{can} \circ \rho)$ it follows that $\rho(y) \in \mathrm{Inn}(N)$. Hence there exists some $x \in N$ such that $(x, y) \in \mathrm{C}_{N\rtimes_\rho H}(N)$. Since $\mathrm{C}_{N\rtimes_\rho H}(N) = \{1\}$ it follows that $y = 1$.

" $\Leftarrow$ " Suppose that $\mathrm{Z}(N) = \{1\}$ and $\mathrm{can} \circ \rho$ is injective. For $(x, y) \in \mathrm{C}_{N\rtimes_\rho H}(N)$ it follows that $\rho(y) \in \mathrm{Inn}(N)$. Hence $y \in \ker(\mathrm{can} \circ \rho) = \{1\}$. Now since $\rho(y) = \rho(1) = \mathrm{id}$ it follows for all $a \in N$ that $\rho(y)(a) = x^{-1}ax = a$. But $\mathrm{Z}(N) = \{1\}$ hence $x = 1$. □

Proposition 7.9. *Let S be a simple non-abelian group, for an index set I let $k = \mathbb{Q}(X_i : i \in I)$ and assume that $|S| \le |I|$. Then $G := \mathrm{PSL}(2, k) \rtimes_\rho S$ is perfect and $\mathrm{Z}(G) = \{1\}$.*

Proof. By [5, p. 45-46] it follows that $\mathrm{PSL}(2, k)$ is simple and therefore G admits a composition series

$$\{1\} \lhd \mathrm{PSL}(2, k) \lhd G.$$

Since G' is a normal subgroup of G it follows that G' is contained in a composition series of G. Now by the Jordan-Hölder Theorem it follows that G/G' is isomorphic to $\mathrm{PSL}(2, k)$, S or $\{1\}$. Since G/G' is abelian it follows that G is perfect. Furthermore $\mathrm{Z}(G)$ is normal in G. It now follows that $\mathrm{Z}(G)$ is isomorphic to G, $\mathrm{PSL}(2, k)$, S or $\{1\}$. Since $\mathrm{Z}(G)$ is abelian it follows that $\mathrm{Z}(G) = \{1\}$. □

Theorem 7.10. *Let S be a non-abelian simple group and for an index set I let $k = \mathbb{Q}(X_i : i \in I)$. We assume that $\#S \le \#I$ and S is not isomorphic to* $\mathrm{PSL}(2, k)$. *For φ defined in Proposition 7.6 and ϕ defined in Lemma 7.7, let $\rho := \varphi \circ \phi$. Then $G :=$* $\mathrm{PSL}(2, k) \rtimes_\rho S$ *is a squarefree group but G does not have the Schreier property and does not satisfy* (2) *of Theorem 6.6.*

Proof. From the proof of Proposition 7.9 it is clear that G admits a composition series. Furthermore since S is not isomorphic to $\mathrm{PSL}(2, k)$ it follows that the composition factors of G are pairwise non-isomorphic. Hence G is squarefree. Now assume that G is a Schreier group. Then by Proposition 7.9 it follows that $G''/\mathrm{Z}(G'') = G$. Now since G is squarefree it follows by Theorem 6.6 that there exists an isomorphism $\mu : \mathrm{PSL}(2, k) \rtimes_\rho S \to S_1 \times S_2$ for S_1, S_2 non-abelian simple groups. Note that $\mathrm{PSL}(2, k) \rtimes_\rho \{1\}$ is normal in G. By Proposition 4.3 it follows that G has precisely 4 normal subgroups. Since $\mathrm{PSL}(2, k) \rtimes_\rho \{1\}$ is a proper non-trivial normal subgroup of G we can assume without loss of generality that $\mu(\mathrm{PSL}(2, k) \rtimes_\rho \{1\}) = S_1 \times \{1\}$. Consider the following commutative diagram:

$$\begin{array}{ccc} S & \overset{\psi\circ\phi}{\rightarrowtail} & \mathrm{Out}(\mathrm{PSL}(2,k)) \\ {\scriptstyle\phi}\downarrow & \overset{\psi}{\nearrow} & \uparrow{\scriptstyle\mathrm{can}} \\ \mathrm{Aut}(k) & \xrightarrow{\varphi} & \mathrm{Aut}(\mathrm{PSL}(2,k)) \end{array}$$

By Proposition 7.6 it follows that

$$\mathrm{can} \circ \rho = \mathrm{can} \circ (\varphi \circ \phi) = (\mathrm{can} \circ \varphi) \circ \phi = \psi \circ \phi$$

is injective and by Proposition 7.9 it follows that $\mathrm{Z}(G) = \{1\}$. Therefore by Proposition 7.8 it follows that

$$\mu \left(\mathrm{C}_G \left(\mathrm{PSL}(2, k) \rtimes_\rho \{1\}\right)\right) = \mu\left(\{1\}\right) = \{1\}.$$

But also

$$\mu \left(\mathrm{C}_G \left(\mathrm{PSL}(2, k) \rtimes_\rho \{1\}\right)\right) = \mathrm{C}_{S_1 \times S_2}(S_1 \times \{1\}) = \{1\} \times S_2$$

which is a contradiction. □

Example 7.11. Take $S = \mathrm{A}_5$ the alternating group of order 60 and let I be an index such that $\#I = 60$. Then $\mathrm{PSL}(2, \mathbb{Q}(X_i : i \in I)) \rtimes_\rho \mathrm{A}_5$ is a squarefree group that does not satisfy the Schreier property and does not satisfy (2) of Theorem 6.6.

7.3 Extending the series

Let G be a squarefree group that does not have the Schreier property and does not satisfy (2) of Theorem 6.6. In Theorem 7.10 we saw that such group exists with length 2. The reader may wonder if such a construction is possible for any length bigger than 1. The answer is yes and we will show how to build such a group.

Lemma 7.12. *Let G be a group and let $N \lhd G$ be simple. If $\mathrm{C}_G(N) = \{1\}$, then for any normal subgroup $M \lhd G$ it follows that $M = \{1\}$ or $N \subset M$.*

Proof. We have that $M \cap N$ is a normal subgroup of N. Since N is simple it follows that $M \cap N = N$ or $M \cap N = \{1\}$. The first case implies that $N \subset M$. For the second case we have by Lemma 2.4 that $[M, N] = \{1\}$. Hence $M \subset \mathrm{C}_G(N) = \{1\}$. □

Lemma 7.13. *Let G be a group and let $N \lhd G$ be simple. If $\mathrm{C}_G(N) = \{1\}$, then N is contained in every composition series of G.*

Proof. If G does not have a composition series, then we are immediately done. Now suppose that G has a composition series

$$\{1\} = G_t \subset G_{t-1} \subset \cdots \subset G_1 \subset G_0 = G.$$

We proceed by induction on $\ell(G)$. If $\ell(G) = 1$, then $G = N$. Next assume $\ell(G) > 1$ and for every subgroup $H \subset G$ with $N \subset H$, $\ell(H) < \ell(G)$ and $\mathrm{C}_H(N) = \{1\}$ assume that N is contained in any composition series of H. We have that $\ell(G_1) = t - 1$ thus $G_1 \neq \{1\}$. Hence by Lemma 7.12 it follows that $N \subset G_1$. Furthermore $\mathrm{C}_{G_1}(N) \subset \mathrm{C}_G(N) = \{1\}$. Hence by the induction hypothesis it follows that N is contained in every composition series of G_1. We conclude that N is contained in every composition series of G. □

Proposition 7.14. *Let $t \in \mathbb{Z}_{>0}$ and let $I_1, I_2, \ldots, I_t$ be infinite index sets such that $\#I_1 < \#I_2 < \cdots < \#I_t$. We define $k_i := \mathbb{Q}(X_j : j \in I_i)$ for $1 \leq i \leq t$. Then there exists a series $\Omega_1, \Omega_2, \ldots, \Omega_t$ of groups and for each $0 < i < t$ a group homomorphism $\eta_i : \Omega_i \to \mathrm{Aut}(\mathrm{PSL}(2, k_{i+1}))$ such that:*

1. *$\Omega_1 = \mathrm{PSL}(2, k_1)$;*
2. *$\Omega_{i+1} = \mathrm{PSL}(2, k_{i+1})) \rtimes_{\eta_i} \Omega_i$ for $0 < i < t$;*
3. *for all $0 < i \leq t$ we have that Ω_i is squarefree, $\ell(\Omega_i) = i$, $\#\Omega_i = \#I_i$ and Ω_i has precisely one composition series.*

Proof. We proceed by induction on t. If $t = 1$, then $\Omega_1 = \mathrm{PSL}(2, k_1)$. Note that Ω_1 is simple and therefore Ω_1 is a squarefree group and has a unique composition series. Furthermore $\ell(\Omega_1) = 1$ and $\#\Omega_1 = \#I_1$. We assume that $t > 1$ and for any $0 < i < t$ we assume that Ω_i is squarefree, $\ell(\Omega_i) = i$, $\#\Omega_i = \#I_i$ and Ω_i has precisely one composition series. By the induction hypothesis it follows that $\#\Omega_{t-1} = \#I_{t-1} < \#I_t$. Hence by Lemma 7.7 there exists an injective group homomorphism $\phi : \Omega_{t-1} \to \mathrm{Aut}(k_t)$. Furthermore by Proposition 7.6 there exists a map $\varphi : \mathrm{Aut}(k_t) \to \mathrm{Aut}(\mathrm{PSL}(2, k_t))$ such that the composite map $\psi := \mathrm{can} \circ \varphi : \mathrm{Aut}(k_t) \to \mathrm{Out}(\mathrm{PSL}(2, k_t))$ is injective. The following commutative diagram summarizes our observations:

$$
\begin{array}{ccc}
\Omega_{t-1} & \xrightarrow{\psi\circ\phi} & \mathrm{Out}(\mathrm{PSL}(2,k_t)) \\
\downarrow \phi & \nearrow \psi & \uparrow \mathrm{can} \\
\mathrm{Aut}(k_t) & \xrightarrow{\varphi} & \mathrm{Aut}(\mathrm{PSL}(2,k_t))
\end{array}
$$

Now let $\eta_{t-1} = \varphi \circ \phi$. Then it follows that $\mathrm{can} \circ \eta_{t-1} = \psi \circ \phi$ is injective. We have that $\mathrm{PSL}(2, k_t) \rtimes_{\eta_{t-1}} \Omega_{t-1} = \Omega_t$ and $\mathrm{Z}(\mathrm{PSL}(2, k_t)) = \{1\}$. Hence by Proposition 7.8 it follows that $\mathrm{C}_{\Omega_t}(\mathrm{PSL}(2, k_t)) = \{1\}$. Now by Lemma 7.13 it follows that $\mathrm{PSL}(2, k_t)$ is contained in every composition series of Ω_t. Furthermore by the induction hypothesis $\Omega_t/\mathrm{PSL}(2, k_t) \cong \Omega_{t-1}$ has a unique composition series. Hence Ω_t has a unique composition series with $\ell(\Omega_t) = t$. Note that the composition factors of Ω_t are $\mathrm{PSL}(2, k_i)$ for $0 < i \leq t$. Furthermore for all $0 < i, j \leq t$ with $i \neq j$ it follows that $\#\mathrm{PSL}(2, k_i) \neq \#\mathrm{PSL}(2, k_j)$. We conclude that Ω_t is squarefree and $\#\Omega_t = \#I_t$. □

Theorem 7.15. *Let $t \in \mathbb{Z}_{>1}$ and let Ω_t be as in Proposition 7.14. We denote*

$$\Omega_t := \mathop{\rtimes}_{i=t}^{1} \mathrm{PSL}(2, k_i).$$

Then Ω_t is a squarefree group but Ω_t is not a Schreier group and does not satisfy (2) *of Theorem 6.6.*

Proof. By Proposition 7.14 it follows that Ω_t is squarefree and has a unique composition series. It follows by Lemma 2.18 that Ω_t' and $\mathrm{Z}(\Omega_t)$ are contained in the composition series of Ω_t. Since $\mathrm{Z}(\Omega_t)$ and Ω_t/Ω_t' are abelian it follows that $\mathrm{Z}(\Omega_t) = \{1\}$ and $\Omega_t/\Omega_t' = \{1\}$. Therefore $\Omega_t''/\mathrm{Z}(\Omega_t'') = \Omega_t$. Now suppose that Ω_t is a Schreier group. Then by Theorem 6.6 it follows that $\Omega_t = \prod_{i=1}^{t} S_i$ for S_i non-abelian and simple and $S_i \not\cong S_j$ for $0 < i, j \leq t$, $i \neq j$. It now follows that Ω_t has $t!$ composition series. This is a contradiction. □

8 References

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