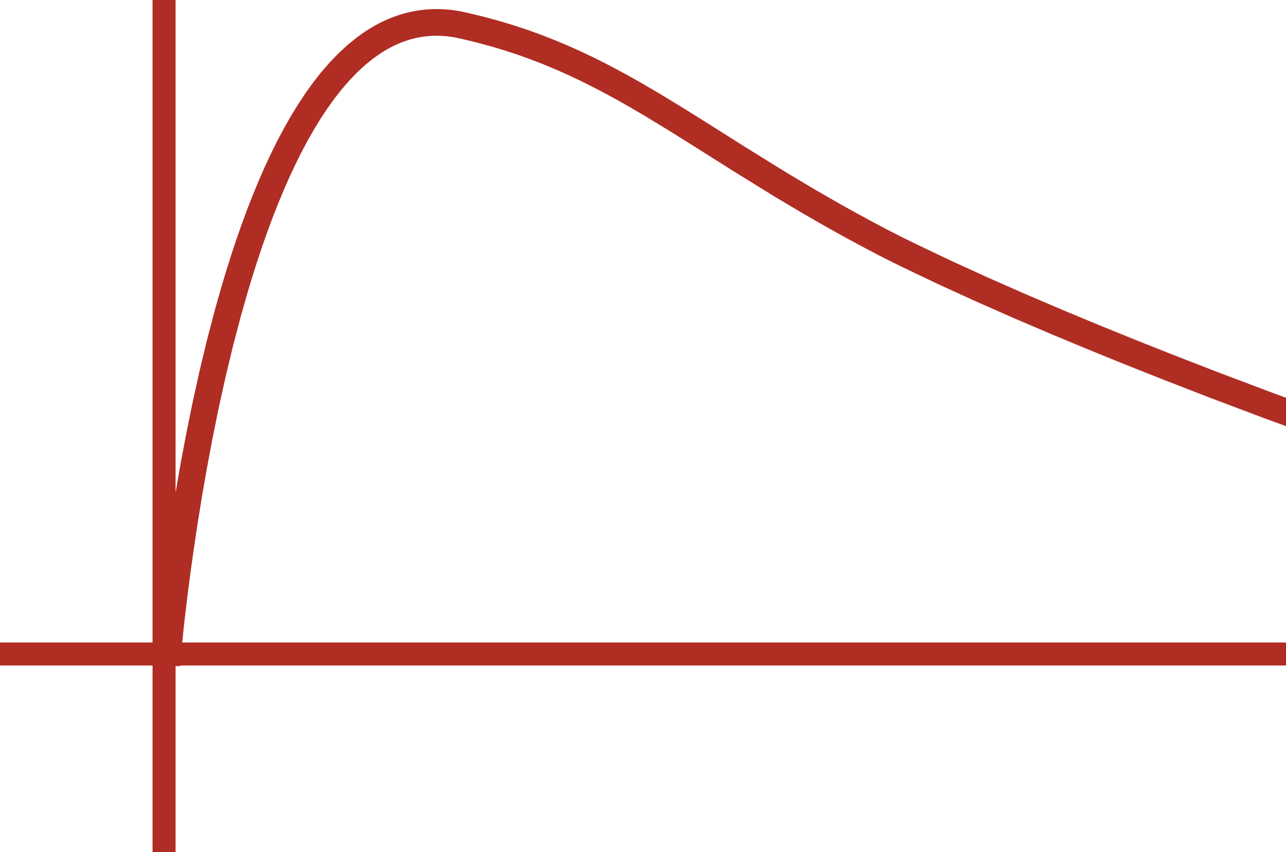
GRAPHING FORMULAS BY HAND TO PROMOTE SYMBOL SENSE

Becoming friends with algebraic formulas

PETER KOP



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PREFACE

For thirty years I have been involved in mathematics education in the Netherlands in several roles: as a teacher, mainly in upper secondary school (grades 10 to 12), participating in two curriculum pilots, as a constructor of national exams, as a teacher educator, as a board member of the Dutch Society of Math Teachers, and as a member of Committee innovation national math curriculum (Commissie Toekomst Wiskunde Onderwijs, cTWO). In all these different contexts algebra education was discussed: what algebra should students learn and how can they learn this? Below I sketch my personal ideas and experiences teaching algebra and my motivation for the research described in this dissertation.

At the start of this century, a radical innovation in upper secondary education was introduced in the Netherlands: at the same time, a new curriculum (Tweede Fase) and a new education concept (studiehuis) were introduced. Along these innovations, emphasis was placed on learning to learn and on (general) skills. In the mathematics curriculum and lessons the graphic calculator was introduced as a permanently available tool, and during exams a formula-card was introduced, containing information about rules for power and logarithm calculations, and a formula for a linear approximation of a function (tangent). Apart from the regular exam problems, students had to tackle larger mathematical tasks, using their problem solving skills.

The graphic calculator could be used e.g. to solve equations via graphs and/or a solver app which seemed to make by hand activities in algebra dispensable. The graphic calculator could be used as a black-box. As a consequence, structures of equations were hardly studied by the students, and the content of the algebra lessons changed.

Before long concerns were expressed by teachers and other stakeholders who realized that effective use of a graphic calculator also requires algebraic skills such as seeing through the structure of formulas and recognizing important characteristics. By then, universities and students had started to complain that students had not been adequately prepared at school and lacked algebraic skills. Universities introduced pen and paper basic skills tests (including solving simple equations and manipulations of algebraic expressions). Prospective students scored very poorly on these tests. To improve the students' algebraic basic skills, many universities opted for explicit and exhaustive practice of basic skills by means of many back-to-back assignments, including simple fraction manipulation, expanding brackets, factorizing, etc. Increasingly more emphasis was put on these algebraic basic skills in secondary school math textbooks. Others, for example Wijers and Kemme (2000), stressed the importance of general algebraic skills like interpreting formulas, relating graph, table and formula, mathematical modeling, that are needed to solve more complex algebraic problems, and pointed out that algebra had to be meaningful for the students.

During this period, my students practiced by hand skills by working on large problems like the historic problem of l 'Hôpital (Drijvers, 1996) in which the minimum of the function $y = \sqrt{0.4^2 - (1-x)^2} + 1 - \sqrt{-0.84 + 2x}$ had to be calculated, and calculating the minimum of $y = ax + b/x$, both with pen and paper, without the graphic calculator. To prepare students for the abovementioned university basic skills, I presented them the university pen and paper basic skill tests. They were able to solve the abovementioned large problems and performed well during the national math exams, but, to my surprise, had trouble with these basic skills tests.

From then on, I decided to pay more attention to by hand algebra activities and found that many students had difficulties learning effective and efficient methods to solve equations by hand, an important aspect of algebraic basic skills. Through ideas from cognitive psychology, I was encouraged to study expert behavior in solving equations by hand: what do they pay attention to when solving equations? Through introspection and interviews with my high-achieving students, who were proficient in this domain, we found that these expert-students use a limited number of categories of equations and could describe these categories (Drijvers & Kop, 2012). I used these categories successfully in my teaching algebra in secondary school and in algebraic courses for prospective university students.

Since the mid-10s, the need for a repertoire of algebraic basic skills which can be performed by hand is currently endorsed by everyone. The Dutch National exams require more algebraic skills by hand, students in the Netherlands have improved basic skills, and the university basic skill tests have silently disappeared. However, this shift towards basic skills had as a consequence that less attention was paid to students' abilities to see structure in algebraic formulas and their reasoning in algebra (Turşucu, Spandaw & de Vries; 2018; Van Stiphout, Drijvers, & Gravemeijer, 2013). At the beginnings of 2010's, the national curriculum was changed again and I participated in the Committee cTWO that formulated new standards. The focus was on extra algebraic basic skills, and therefore analytical geometry was introduced. I was involved in writing the separate algebra section that was added to the curriculum in which algebraic skills were described. At the same time, the phrase “calculate exactly” was introduced to indicate that the graphic calculator could not be used in solving the problem.

The school textbooks have increasingly started earlier with algebraic basic skills and the algebra results of the National exams seemed to improve. However, students continue to have problems with algebra: it is very abstract for them and not very meaningful. Many students seem to use memorized tricks, and hardly learn to read through formulas.
Since 2002, I have also been working as a math teacher educator at ICLON-Leiden University Graduate School of Teaching. In that role, I contributed several chapters on algebra to the Handboek Wiskundedidactiek (Handbook Mathematic Didactics), organized by Anne van Streun (Drijvers, Van Streun, & Zwaneveld, 2012). These writing experiences stimulated me to take the opportunity to start a PhD when it was offered by ICLON, which made it possible to investigate how students’ abilities to read through algebraic formulas and to give meaning to them might be promoted.

CHAPTER 1

General Introduction

1.1 Introduction

Algebra is difficult for many secondary school students, and even beyond secondary school (Kieran, 2006). In literature again and again (e.g. Arcavi, Drijvers, & Stacey, 2017; Chazan & Yerushalmy, 2003; Drijvers, Goddijn, & Kindt, 2011; Kieran, 2006; Arcavi, 1994; Ayalon, Watson, & Lerman, 2015; Hoch & Dreyfus, 2005, 2010; Oehrtman, Carlson, & Thompson, 2008) it is found that:

- for many secondary school students' algebraic formulas with their symbols are very abstract
- students have difficulties to give meaning to algebraic formulas
- students have serious cognitive and affective difficulties with algebra, resulting in a lack of confidence to engage in algebra
- algebra is often taught through series of similar exercises, with a focus on basic skills with algebraic calculations.

On all educational levels, students have problems reading through algebraic formulas, that is, to see a formula as a whole rather than a concatenation of letters, and to recognize its global characteristics: they lack symbol sense (Arcavi, 1994). Arcavi introduced the concept of symbol sense as "an intuitive feel for when to call on symbols in the process of solving a problem, and conversely, when to abandon a symbolic treatment for better tools". Drijvers et al. (2011) see symbol sense as complementary to basic skills. Basic skills in algebra are about procedural work, with a local focus and an emphasis on algebraic calculations. Symbol sense is about taking a global view, adopting a strategic approach and algebraic reasoning, and forms a compass for basic skills. For example, expanding brackets is a basic skill, but whether it is efficient to expand brackets in a problem situation is a matter of symbol sense. Symbol sense is especially important if a task is not recognized as a standard algebraic task and basic skills cannot be used immediately. In such situations, symbol sense is needed to know how basic skills can be used and which. Therefore, symbol sense is indispensable in solving non-routine algebraic tasks, and vice versa, students' performances in non-routine tasks is a measure of their symbol sense.

Symbol sense is very broad and is involved in three phases of the problem-solving cycle. Pierce and Stacey (2004) used the concept "algebraic insight" to capture the symbol sense involved in the solving phase when using computer algebra (CAS), that is, proceeding from the mathematical problem to the mathematical solution (see Figure 1.1).

Real world problem	**Formulate** →	**Mathematical problem**
Check ↑	⇧ **Symbol sense** ⇨ ⇩	**Solve** ↓
Real world solution	← **Interpret**	**Mathematical solution**

Figure 1.1 Symbol sense in problem-solving cycle, based on Pierce & Stacey (2004)

Algebraic insight has to do with the ability to plan, monitor, estimate and interpret algebraic calculations, and has two aspects: algebraic expectation and ability to link representations. Algebraic expectation is about skills to scan expressions for clues that allow one to see and predict patterns and make sense of symbolic operations. A more detailed description is shown in Table 1.1.

Table 1.1 Algebraic insight framework (Pierce and Stacey, 2004)

1	Algebraic expectation
1.1	Recognition of conventions and basic properties
1.1.1.	know meaning of symbols
1.1.2.	know order of operations
1.1.3.	know properties of operations
1.2.	Identification of structure
1.2.1.	identify objects
1.2.2.	identify strategic groups of components
1.2.3.	recognize simple factors
1.3.	Identification of key features
1.3.1.	identify form
1.3.2.	identify dominant term
1.3.3.	link form with solution type
2	Ability to link representations
2.1	Linking of symbolic and graphic representations
2.1.1	link form with shape
2.1.2	link key features with likely positions
2.1.3	link key features with intercepts and asymptotes

In our research we aimed at the development of students' abilities to read though algebraic formulas and to make sense of them. In their algebraic insight, Pierce and Stacey (2004) focused on interpreting and making sense of algebraic calculations that are performed via

CAS, and they included manipulations of formulas, for instance, to determine equivalence of formulas. As our research was not on manipulating algebraic formulas, but exclusively on reading through algebraic formulas and making sense of them, we focused on a subset of algebraic insight. Therefore, we use the term insight into algebraic formulas: to identify the structure of a formula and its components, and to reason with and about formulas. Identifying structure in algebra includes abilities such as seeing an algebraic expression as an entity, recognizing the expression as a previously met structure, dividing the entity into sub-structures, and recognizing the connection between structures (Hoch & Dreyfus, 2010). Teaching symbol sense is not straightforward (Arcavi et al., 2017; Hoch & Dreyfus, 2005). In this thesis we investigated how to promote this insight into algebraic formulas for students in grades 11 and 12.

1.2 Using graphs to learn about formulas

Many studies have suggested how students might learn about linear formulas and make sense of them by linking them to realistic contexts. In lower secondary school it is easy to link linear and exponential formulas to realistic contexts, but for more complex formulas in upper secondary school, like logarithmic, root, rational functions, and compositions of functions, the link to realistic contexts is in general difficult. There is less research how the students in upper secondary school might learn to develop insight into these more complex formulas. Besides linking formulas to realistic contexts, Kieran (2006) and Radford (2004) have suggested using multiple representations to make sense of formulas. A formula is one of the representations of a mathematical function, besides others like a table, a graph, a verbal description. A mathematical object like a function can only be studied through its representations. Different representations give different information about a function (Arzarello, Bazzini, & Chiappini, 2001). Formulas stress the input-output dependency, whereas graphs give a Gestalt-view of the function, visualizing the “story” a function tells in a single picture. Interpreting graphs seems easier than interpreting algebraic formulas for students, as graphs seem to be more concrete for them.

In mathematics education, to use graphing tools such as graphic calculators for learning about functions and their multiple representations is recommended (Hennessy, Fung, & Scanlon, 2001; Kieran & Drijvers, 2006; Heid, Thomas, & Zbiek, 2013; Philipp, Martin, & Richgels, 1993; Yerushalmy & Gafni, 1992). However, Goldenberg (1988) found that students established the connection between formula and graph more effectively when they did graph by hand than if they only performed computer graphing. Others have confirmed the need for pen and paper activities when learning about formulas (Kieran & Drijvers, 2006). Therefore, we focused on graphing formulas by hand, without technology. In this thesis we refer to this by *graphing formulas*. In the past, graphing a formula was a time-consuming goal in itself. Via a fixed step-by-step plan, a function was investigated by calculating its domain, zeroes, extreme values (via the derivative), and asymptotes. This approach caused students to focus on the many calculations, and not to reason about the functions. As our aim is to promote students' insight into algebraic formulas, our approach of graphing formulas does not focus on calculations and on detailed graphing but on reading through formulas, reasoning and rough sketches of a graph.

1.3 Students' difficulties learning about formulas

Using graphing formulas to promote insight into formulas might address in a natural and integrated way several aspects that seem problematic in learning about functions.

First, mathematical objects like functions are not directly accessible as physical objects. Only through representations and combining the information obtained through different representations can one understand a function and the rich concept image of the function (Thomas, Wilson, Corballis, Lim, & Yoon, 2010; Tall & Vinner, 1981). Thus, the translation from one representation of a concept to another is, like in graphing formulas, at the core of doing and understanding mathematics (Duval, 2006; Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009; Lesh, 1999; Thomas & Hong, 2001).

Second, students are often found to have difficulties with the so-called process—object character of a function, that is, seeing a function both as an input-output machine and as an object, which can be used e.g. to reason about and to categorize (Ayalon et al., 2015; Breidenbach, Dubinsky, Hawks, & Nichols; 1992; Gray & Tall, 1994; Moschkovich, Schoenfeld, & Arcavi, 1993; Oehrtman et al., 2008; Sfard, 1991). Formulas stress the function's process character, but graphs appeal to a Gestalt-view, and stress the object character (Kieran, 2006; Moschkovich et al., 1993; Schwartz & Yerushalmy, 1992).

Third, graphing formulas is related to covariational reasoning. Students have difficulties with this kind of reasoning. Covariational reasoning is the ability to coordinate an image of two varying quantities and to note how they change in relation to each other (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) and is found to be essential to understand major concepts of calculus: functions, limits, derivatives, rates of change, concavity, inflection points, and their real world interpretations (Carlson et al., 2002; Oehrtman et al., 2008). Covariational reasoning is often used in realistic contexts, but it is also used with algebraic functions when, “imagining running through all input-output pairs simultaneously and so reason about how a function is acting on an entire interval of input values” (Carlson et al., 2002). Carlson, Madison, and West (2015) found that in an exam only 37% of the university students were able to select the correct graph (out of five alternatives) of $f(x) = 1/(x-2)^2$, indicating, according to the authors, that many students were not able to reason “as the value of x gets larger the value of y decreases, and as the value of x approaches 2, the value of y increases.”

Fourth, students have difficulties to recognize the structure of formulas. Ernest (1990) suggested to construct a syntactical tree: via an iterative procedure, algebraic expressions are decomposed into meaningful parts (building blocks) by identifying the main operator of the expression.

In sum, through graphing formulas attention is paid to switching between representations of functions, to the process—object character of functions, to covariational reasoning, and to the structure of formulas and their components.

1.4 Research questions

To engage in algebra, one needs a combination of basic skills and symbol sense. In regular education there is an overemphasis on basic skills. It has been acknowledged that it is hard to teach symbol sense (Arcavi et al., 2017; Hoch & Dreyfus, 2005). In this research we focused on one aspect of symbol sense, that is, insight into formulas. Insight has been defined as the ability to recognize the structure of a formula and its key features and to reason with and about a formula. To teach this aspect of symbol sense, we chose graphing formulas by hand. Graphing formulas by hand requires reading through formulas and includes many other aspects that students find difficult when learning about functions. This led to the overall research question of this thesis:

How can teaching graphing formulas foster grade 11 and 12 students’ insight into formulas and their symbol sense to solve non-routine algebraic problems?

1.5 Outline of the thesis

To investigate this main question, we designed an intervention in which a group of 21 students were taught how to graph formulas. However, it was not clear what knowledge and which skills are needed to graph formulas. In studies 1 and 2, we studied expert behavior in graphing formulas and their recognition processes. In study 3, the intervention, based on expert strategies in graphing formulas, was designed and tested. In study 4 we investigated whether there is a positive relation between students' abilities to graph formulas and their abilities to solve non-routine algebraic problems with symbol sense. In chapter 6, we summarize the findings of the studies presented and the practical and scientific implications.

1.6 Characterization of studies 1–4

Study 1, described in chapter 2, was about identifying a framework for graphing formulas from expert strategies. Although graphing formulas is a well-described task, it can be complex because of the large variety of functions that may be involved. To investigate what is needed for such a complex task, to examine expert behavior has been recommended (Schoenfeld, 1978; Kirschner & Van Merriënboer, 2008). In expertise research it has been found that experts, compared to novices, have more structured knowledge, which enable them to make a more sophisticated problem representation and reach higher levels of recognition, and to search more efficiently in a problem space (Chi, Feltovich, & Glaser, 1981; Chi, Glaser, & Rees, 1982; Chi, 2011; De Groot, 1965; De Groot, Gobet, & Jongman, 1996; Gobet, 1998). Berliner and Ebeling (1989) formulated a model in which performance is a function of two variables: recognition and heuristic search. In this model, the degree of recognition determines the problem space and, as a consequence, the heuristic search. Thus, although expertise is described in terms of recognition and heuristic search in the literature (Chi, 2011; Gobet, 1998; Gobet & Simon, 1996), it appears this interplay has never been used in designing concrete teaching. We formulated a two-dimensional framework to describe strategies in graphing formulas, using levels of recognition and at each level of recognition, heuristics, that is, reasoning with and about formulas, to graph formulas. The levels of recognition in this framework reflect the different levels of awareness that have been formulated by Mason (2003): from complete recognition and instantly knowing the graph, to decomposing the formula into manageable sub-formulas, to perceiving graph properties, to no recognition at all and only calculating some points. We had two research questions: Does the framework describe strategies in graphing formulas appropriately and discriminatively? Which strategies do experts use in tasks graphing formulas? Five experts and three secondary-school math teachers were asked to solve two complex graphing tasks while thinking aloud.

The second study, in chapter 3, was about unraveling experts' recognition processes. Our research questions were: Can we describe experts' repertoires of instant graphable formulas (IGFs) using categories of function families? What do experts attend to when linking formulas and graphs of IGFs, described in terms of prototype, attribute, and part-whole reasoning?

Three different tasks were developed to elicit the experts' repertoires of IGFs and to explore the experts' recognition processes: a card-sorting task, a matching task, and a thinking-aloud multiple-choice task. The tasks were administered to the same five experts from study 1. The participants' categorizations of the card-sorting were compared to an expert categorization. The data analysis of the multiple-choice task was based on Barsalou (1992) and Schwarz and Hershkowitz (1999), using prototype, attribute, and part-whole reasoning.

The third study, in chapter 4, we investigated how to teach grade 11 students expertise in graphing formulas, that is, using a combination of recognition and qualitative reasoning to graph formulas. The research question addressed was: How can grade 11 students' insight into algebraic formulas be promoted through graphing formulas? An intervention consisting of a series of five 90 minutes lessons was designed, using principles of teaching complex skills and the meta-heuristic "questioning the formula". The teaching focused on a repertoire of basic function families with their characteristics and on qualitative reasoning using prototypes, graph features, and exploration of parts of a graph, like infinity behavior. A group of 21 grade 11 students were involved in the intervention and made a pre, post, and retention tests, and filled in as well a post-intervention questionnaire. During the pre and post tests, six students were asked to think aloud.

The fourth study, in chapter 5, explored the relation between students' graphing abilities and their symbol sense abilities to solve non-routine algebraic tasks. We investigated whether students might be able to use insight learned in the domain of graphing formulas, in broader domains of algebra, with problems like: How many solutions does this equation have? What y-values can this formula have?

We limited the algebraic problems to those that can be solved with graphs and reasoning. Besides the symbol sense involved in graphing formulas, the students needed also another aspect of symbol sense, namely, to abandon the symbolic representation, and to use graphs and/or reasoning, instead of starting calculations. This led to the fourth research question: How do grade

12 students' abilities to graph formulas by hand relate to their use of symbol sense while solving non-routine algebra tasks? We formulated two sub-questions: To what extent are students' graphing formulas by hand abilities positively correlated to their abilities to solve algebraic tasks with symbol sense? Is students' use of symbol sense in graphing formulas similar or different from their use of symbol sense in solving non-routine algebraic tasks? A symbol sense test was administered to a group of 114 grade 12 students, including 21 students who had participated in a previous intervention described in study 3. The test consisted of 8 graphing tasks and 12 non-routine algebraic tasks, which could be solved by graphing and reasoning. The results of the written test were graded, and the symbol sense use was analyzed using four categories: blank, calculations, making a graph, recognition and reasoning. To get a more detailed picture of students' symbol sense, six students, all involved in the intervention of the third study, were asked to think aloud during the symbol sense test.

CHAPTER 2

Identifying a framework for graphing formulas from expert strategies

This chapter is based on: Kop, P. M., Janssen, F. J., Drijvers, P. H., Veenman, M.V., & Van Driel, J.H. (2015). Identifying a framework for graphing formulas from expert strategies. *The Journal of Mathematical Behavior*, *39*, 121–134.

Abstract

It is still largely unknown what are effective and efficient strategies for graphing formulas with paper and pencil, without the help of graphing tools. We here propose a two-dimensional framework to describe the various strategies for graphing formulas with recognition and heuristics as dimensions. Five experts and three secondary-school math teachers were asked to solve two complex graphing tasks. The results show that the framework can be used to describe formula graphing strategies and allows for differentiation between individuals. Experts used various strategies when graphing formulas: some focused on their repertoire of formulas they can instantly visualize by graphs; others relied on strong heuristics, such as qualitative reasoning. Our exploratory study is a first step towards further research in this area, with the ultimate aim of improving students' skills in reading and graphing formulas.

2.1 Introduction

Students often have difficulties with algebra, in particular giving meaning to and grasping the structure of algebraic formulas, and manipulating them (Chazan & Yerushalmy, 2003; Drijvers et al., 2011; Kieran, 2006; Sfard & Linchevski, 1994). Functions can be represented in several forms, such as algebraic formulas and graphs; the latter are more accessible for students than the former (Janvier, 1987; Leinhardt, Zaslavsky, & Stein, 1990; Moschkovich et al., 1993).

A graphical representation gives information on covariation, that is, how the *y*-coordinate (the dependent variable) changes as a result of changes of the *x*-coordinate (the independent variable) (Carlson et al., 2002). A graph shows possible symmetry, intervals of increase or decrease, extreme values, and infinity behavior. In this way, it visualizes the "story" of an algebraic formula. Graphs may help learners to give meaning to algebraic formulas and so make learning algebra easier for them (Eisenberg & Dreyfus, 1994; Kilpatrick & Izsak, 2008; NCTM, 2000; Philipp, Martin, & Richgels, 1993; Yerushalmy & Gafni, 1992).

Graphs are also considered important in problem solving (Polya, 1945; Stylianou & Silver, 2004). In his list of heuristics, Polya (1945) mentions drawing a picture or diagram as one of the first options. Creating and using multiple representations, and switching between them, are important tools in problem solving (Janvier, 1987; NCTM, 2000). Stylianou and Silver (2004) and Stylianou (2002, 2010) found how graphs are used to understand the problem situation, to record information, to explore, and to monitor and evaluate results.

For learning about functions, graphing tools such as graphic calculators are recommended (Drijvers & Doorman, 1996; Drijvers, 2002; Hennessy et al., 2001; Kieran & Drijvers, 2006; Philipp et al., 1993; Schwartz & Yerusalmy, 1992; Yerushalmy & Gafni, 1992). With these tools, graphing formulas seems easy. In the past, constructing a graph was itself a goal or the graph itself an end product. To produce one, many algebraic skills (determining domain, zeroes, derivative, etc.) were employed, along with standard methods requiring multiple algebraic manipulations, which were not straightforward for all learners.

Graphing tools now make it possible to study problems that in the past could not be solved or could be solved only with difficulty. In order to use these tools adequately, however, one must know what aspects of graphs to look for (Philipp et al., 1993). According to Stylianou and Silver (2004) novices experience difficulties in the visual explorations of the graphs they have constructed. They concluded that such explorations are restricted to familiar functions. So, in order to make effective and efficient use of technology, learners should know about graphs representing basic functions, and also should have learned to reason about such graphs (Drijvers, 2002; Eisenberg & Dreyfus, 1994; Stylianou & Silver, 2004).

Learners who do graphing with pen and paper may establish the connection between the algebraic and the graphical representations of a function more effectively than learners who only perform computer graphing (Goldenberg, 1988). In this article, graphing to produce a sketch of a graph with its main characteristics without technological help will be called *graphing formulas*.

Despite earlier research on how to learn and how to teach functions, it is still largely unknown what knowledge and skills are necessary to graph formulas effectively and efficiently. In order to learn more about these, we have identified expert strategies in our research. Experts are expected to know and use more effective and efficient strategies than novices (Chi, 2006, 2011). Hence, the focus of this article will be on determining a suitable framework for formula graphing strategies. With the help of this knowledge base, a professional development trajectory for teachers and teaching material for students may eventually be developed.

2.2 Theory

2.2.1 Aspects of graphing formulas

Functions are at the core of math education. There are several reasons for students' difficulties with the concept. Functions, like other mathematical concepts, are not directly accessible as physical objects. Access to mathematical concepts can only be gained through representations. To understand mathematical concepts, one needs to relate elements of different representations (Janvier, 1987; Kaput, 1998). For functions, these representations are algebraic formulas, graphs, tables, and contexts (Janvier, 1987). These representations have to be combined in order to produce a rich concept image of the function (Thomas et al., 2010; Tall & Vinner, 1981).

The ability to represent concepts, to establish meaningful links between and within representations, and to translate from one representation of a concept to another is at the core of doing and understanding mathematics. Different concepts have been used to refer to this ability: representational flexibility (Nistal et al., 2009), representational fluency (Lesh, 1999), representational versatility (Thomas & Hong, 2001). ‘Representational versatility’ has been defined as the ability to work seamlessly within and between representations and to engage in procedural and conceptual interactions with representations (Thomas et al., 2010). Our research deals with translations between algebraic formulas and graphs, demonstrating representational versatility.

Much research has been done on algebraic and graphical representations and their relations. Students are often found to have difficulties with reading algebraic formulas and the so-called process-object character of a function. For graphing formulas, it is necessary that one can “read” algebraic formulas and deal with the process-object character of a function. These two issues are discussed in the next sections.

2.2.2 Reading algebraic formulas

There are different ways to create meaning for algebraic formulas: from the problem context, from the algebraic structure of the formula, and from its various representations (Kieran, 2006). In order to read an algebraic formula, one has to grasp its structure (Sfard & Linchevski, 1994). In the literature this is called ‘symbol sense’ (Arcavi, 1994). Symbol sense has several aspects, such as the ability to read through algebraic expressions, to see the expression as a whole rather than a concatenation of letters, and to recognize its global characteristics (Arcavi, 1994). Symbol sense enables people to scan an algebraic expression so as to make rough estimates of the patterns that would emerge in numeric or graphical representations (Arcavi, 1994).

A procedure for analyzing the syntactic structure of an expression was formulated by Ernest (1990). A syntactical tree is constructed via an iterative procedure in which the main operator of the expression is identified. The procedure continues until all subexpressions have been given meaning. The decomposition of algebraic expressions into meaningful parts (building blocks) can be considered a heuristic for reading formulas.

Thomas et al. (2010) asked students about their strategies when linking formulas to graphs of linear and quadratic functions. He found that the students tried to imagine the graph and focused on key properties of the functions. Thomas et al.’s findings are consistent with Mason’s (2002, 2004) argument that both perception of and reasoning about functions are driven by specific properties of the representation (Thomas et al., 2010). Mason (2003, 2004) identified levels of attention or awareness, which he used to describe how a person’s attention can shift from staring at the whole (for example an algebraic formula) while hardly knowing how to proceed, to discerning details from which objects and sub-objects can be determined in order to recognize relationships, perceive properties, and grasp the essential structure. For this recognition, students need a repertoire of basic functions, and knowledge of the characteristics of the representations of these functions (Eisenberg & Dreyfus, 1994).

In summary, for reading algebraic formulas recognition of basic functions and symbol sense are important, as well as knowledge of procedures such as decomposing formulas into meaningful sub-formulas.

2.2.3 Process and object perspectives

Students often use only the process perspective of a function, because they see a function as a calculation rule in which *x* and *y* values are linked. Formulas which differ from the process perspective, for instance $y = x + 3$ and $y = 4 + x - 1$, can belong to the same function. From the object perspective, such formulas are considered identical (Schwartz & Yerusalmy, 1992). The object perspective is needed to perform actions on a function, for instance to transform it (Breidenbach et al., 1992) and to classify families of functions.

Students have to be able to assume both a process and an object perspective and to switch between the two (Breidenbach et al., 1992; Gray and Tall, 1994; Oehrtman et al., 2008; Sfard, 1991). Moschkovich et al. (1992) formulated a two-dimensional framework, with representations and perspectives as dimensions, and showed that in problem solving different representations and different perspectives are needed. There is a relation between these two dimensions: the algebraic representation of a function makes its process perspective salient, while the graphical representation suppresses the process perspective and thus helps to make a function more entity-like, i.e., an object (Schwartz & Yerushalmy, 1992; Moschkovich et al., 1992).

Three theories consider the complementary aspects of process and object perspectives: APOS, covariational reasoning, and pointwise and global approach in solving problems with formulas and graphs.

The APOS (action, process, object, schema) theory describes how an object perspective is developed through the encapsulation of processes, and how a schema integrating both perspectives is created (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Breidenbach et al., 1992). A well-developed schema can be seen as a cognitive unit, an element of cognitive structure that in its entirety can be the focus of attention at a given time

(Barnard & Tall, 1997). Such a cognitive unit can be activated as a single step in a thinking process (Crowley & Tall, 1999). This type of schema makes it possible to instantly switch between process and object perspectives.

Covariational reasoning is the ability to coordinate an image of two varying quantities and to note how they change in relation to each other (Carlson et al., 2002). It is essential to understand major concepts of calculus: functions, limits, derivatives, rates of change, concavity, inflection points, and their real-world interpretations (Carlson et al., 2002; Oehrtman et al., 2008). In covariational reasoning, one is able to imagine running through all input-output pairs simultaneously and so to reason about how a function is acting on an entire interval of input values. Such reasoning is not possible operating from a process perspective where each individual computation must be explicitly performed or imagined. Carlson et al. (2002) describe levels of covariational reasoning: from the notion of "y is changing with changes in x", via knowing whether a function increases or decreases, and considerations about the rate of change, to understanding average and instantaneous rates of change, and inflection points.

Even (1998) considers the process and object perspectives in solving problems with formulas and graphs in terms of heuristics. She distinguishes pointwise and global approaches. In the pointwise approach, students plot and read points, whereas in the global approach they focus on the behavior of the function on an interval or in a global way. The global approach is more powerful and gives a better understanding of the relation between formulas and graphs. However, sometimes the pointwise approach is needed to monitor naïve and/or immature interpretations and to construct meaning (Even, 1998).

In summary, in order to graph formulas effectively and efficiently one has to be able to read algebraic formulas and deal with the process and object perspectives. The literature shows that both recognition through schemas and symbol sense are important in this respect. If recognition fails, heuristics are necessary. In our research, we attempted to elucidate which recognition and heuristics are essential for effective and efficient graphing, so as to identify a framework of strategies for graphing formulas. For the necessary background knowledge on how recognition and heuristics are related we consulted the literature on expertise, in which the importance of recognition and heuristics is endorsed.

2.2.4 Expertise

Experts outperform novices when solving problems in their fields of expertise (Chi et al., 1981; Chi et al., 1982; Chi, 2006, 2011). What is the reason for this difference in performance? In order to sketch the main components of expertise and their interrelations, we here provide a short historical review of the changing explanations of expertise during the past forty years (Chi, 2011).

First, experts were believed to have superior search strategies. In expertise research in the 1970s, expertise was often assessed via puzzle-like problems. In order to solve such puzzles participants need general problem-solving strategies rather than much domain-specific knowledge. Within this context, solving a problem is seen as searching for a path in the problem space to connect the problem with the solution. Expertise is defined as the ability to search efficiently and effectively. Therefore, general heuristics have been formulated for mathematical problem solving (Polya, 1945; Marshall, 1995; Schoenfeld, 1985, 1992; Van Streun, 1989). In problem-solving literature, recognition is often mentioned as an all-or-nothing process in the orientation on a problem: either there is recognition or there is not.

Subsequently, it was more structured knowledge that was thought to be the decisive factor determining search strategies. Experts do not necessarily have superior general search strategies, but their knowledge is more effectively structured, as cognitive schemas in the long-term memory (Chase & Simon, 1973). A cognitive schema can be seen as a network with (hierarchically) related concepts, procedures, and strategies (Anderson, 1980; Derry, 1996). To a large extent a person’s cognitive schemas determine what that person "sees" and recognizes in problem situations (Sweller, Merriënboer, & Paas, 2019). Thus, when someone is confronted with a problem situation, different levels of recognition are possible, varying from completely recalling the problem situation and the solution to no recognition at all.

In mathematics, when novices see an algebraic formula, as for instance $y = x^2 + 2x$, they may not recognize it as a polynomial function, but see it as just a procedure: $y = x \cdot x + 2 \cdot x$. Experts seeing $y = x^2 + 2x$, will immediately recognize it as a member of the family $y = ax^2 + bx + c$ and will know that the graph will be a parabola with a minimum. In an expert’s cognitive schema, activated by the algebraic formula, the formula can be linked to a graph and thus can be instantly visualized graphically, whereas other formulas cannot, if the cognitive schema activated by the formula does not have a link to its graph. In formula graphing the set of functions that can be instantly visualized plays an important role. These functions can be seen as units or building blocks for thinking and reasoning in simple as well as in complex situations.

Recently, it is the *representation* of the problem that has come to be seen as the dominant factor accounting for expertise: structured knowledge guides representation, which dictates search strategies. Experts and novices focus on different elements of the problem they are confronted with. In physics, experts look for the underlying principles on which a problem is based, whereas novices look at the superficial surface characteristics (Chi et al., 1981). Experts activate schemas that can provide additional information, strategies, and expectations for further elaboration of the problem representation (Chi et al., 1981). In chess research, two main explanations for expertise are given: the ability to access a rich knowledge database through pattern recognition, and the ability to search efficiently through the problem space via a deliberate heuristic search (De Groot, 1965; De Groot, Gobet, & Jongman, 1996; Gobet, 1997, 1998). Berliner and Ebeling (1989) formulated a model in which performance is a function of two variables, i.e., recognition and search. In this model, different combinations of recognition and search may give the same level of performance, as there is a trade-off between recognition and search. Berliner and Ebeling (1989) concluded that the degree of recognition determines the problem space and, as a consequence, the heuristic search.

In mathematics, when experts have to graph a function such as $y = x^2 + 2x$ their algebraic knowledge will tell them that $y = (x + 1)^2 - 1$ and $y = x(x + 2)$ are equivalent formulas. So, the expert can use all these algebraic representations to acquire detailed information about the graph. Novices faced with the formula $y = x^2 + 2x$ lack the ability to switch to an alternative problem representation. Thus, in problem solving levels of recognition depend on knowledge. Experts' structured knowledge facilitates high levels of recognition, which gives them superior problem representation and efficient heuristic search options. In the next section, we will describe different levels of recognition in graphing formulas and show how in graphing formulas recognition and heuristic search may be related.

2.2.5 Towards a two-dimensional framework

Research on expertise has indicated that recognition guides heuristic searching, i.e., the level of recognition determines the heuristic search. This interplay can result in a two-dimensional framework by which to describe strategies in graphing formulas. Recognition and heuristic search will form the two dimensions of our framework. Below, we will first give an example to illustrate how different levels of recognition allow different heuristic searches. Then we will formulate levels of recognition and of efficient heuristic searching in graphing formulas.

Example: When we have to graph a complex formula, such as $f(x) = x^2\sqrt{8-x} + 2x$, it is difficult to immediately visualize its graph. A heuristic search is needed. A first step can be to consider the domain: $<\leftarrow,8]$. It is possible to decompose the function into two subformulas, $y = x^2\sqrt{8-x}$ and $y = 2x$ (Ernest, 1990). While the second subformula can probably be visualized instantly, the first, $y = x^2\sqrt{8-x}$ probably cannot. Via qualitative reasoning about its infinity behaviour at $-\infty$ (if $x \rightarrow -\infty$ then $y \rightarrow \infty$) and its zeroes 0 and 8, the graph can be sketched (see Figure 2.1). Then the graph for $f(x) = x^2\sqrt{8-x} + 2x$ can be constructed via qualitative reasoning and/or by making a table (Figure 2.2).

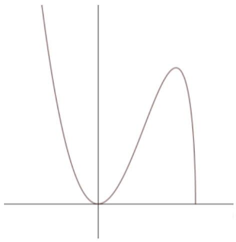


Figure 2.1 Graph of $y = x^2\sqrt{8-x}$

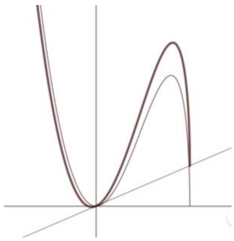


Figure 2.2 Construction of the graph $f(x) = x^2\sqrt{8-x} + 2x$

Another strategy is to reason qualitatively about domain and zeroes of the function $f(x) = x^2\sqrt{8-x} + 2x$, and then to calculate the derivative in order to find the extreme values. Novices may not recognize the structure of the formula and will probably be limited to using a heuristic such as "make a table". This example shows how recognition determines the problem representation and hence the heuristic search: recognition guides heuristic search. In the next section different levels of recognition and of heuristic search are identified.

2.2.6 Levels of recognition and heuristic search

In this section on the different levels of recognition and heuristics used in our framework we will first focus on recognition, and then on heuristic searching in graphing formulas. The literature on reading formulas and expertise mentioned in Theory section 2.2, and the above example, suggest that different levels of recognition are possible: from complete recognition (the graph is instantly known), to decomposing a formula into known subgraphs, to no recognition of the graph. On the basis of the literature, our own reflections, and several pilot interviews before our research, we propose the following six levels of recognition:

- Level A: the graph is immediately recognized. For instance, graphs of $y = -2x + 4$ and $y = x^2$ are instantly visualized.
- Level B: the equation is recognized as a member of a family of which the possible graphs are known. Only a brief analysis is needed to graph the formula. For instance, $y = 4 \cdot 0.75^x + 3$ is recognized as belonging to the family of decreasing exponential functions; $y = -x^4 + 6x^2$ is described as a polynomial function of degree 4, so its graph has an M or Λ form because of the negative main coefficient.
- Level C: the formula is split into sub-formulas that can be instantly visualized. For instance, $y = x + 4/x$ is decomposed into $y = x$ and $y = 4/x$, or $y = 2x\sqrt{x+6}$ into $y = 2x$ and $y = \sqrt{x+6}$
- Level D: characteristic aspects of the graph are recognized but the rest of the graph is not. For instance, the graph of $y = x + 4/x$ is described as 'having a slanted asymptote $y = x$ and a vertical asymptote $x = 0$', but no other features of the graph are described.
- Level E: the graph is not even partly recognized, but the participant is able to use the algebraic formula for deliberate exploration of the graph. For instance, the formula $y = x^2/(x^2 + 2)$ is analyzed by qualitative reasoning: domain is $\mathbb{R}$, zero at $x = 0$; the graph is symmetrical; if $x \to \infty$ then $y \to 1$ (infinity behavior), so $y = 1$ is a horizontal asymptote; the graph increases for positive x values.
- Level F: the graph is not recognized, and neither are any features of the algebraic formula. At this level one is restricted to using a standard repertoire to find characteristics of the graph, i.e., domain, zeroes, and extreme values via the derivative, or making a table with random x values.

These levels of recognition can be linked to Mason's levels of attention (Mason, 2003) and Thomas and Hong's model of interaction with representations (Thomas & Hong, 2001). The latter gives a hierarchy of observations of a representation: from surface observation, via noting properties, to actions on the representation in order to obtain further information or understanding of a concept (Thomas & Hong, 2001). Our levels of recognition can be linked to Mason's levels of attention as follows: gazing at the whole (an algebraic formula) while hardly being aware how to proceed can be related to level F, discerning details to level E, recognizing relationships between different details, perceiving properties, and seeing the essential structure to levels C, B or A.

Expertise research shows that experts use more efficient heuristics than novices do. These efficient heuristics are called ‘strong’. The stronger the heuristics, the more characteristics of the problem situation and domain-specific knowledge will be used, which results in faster problem solving. When graphing formulas these stronger heuristics will result in more information about the whole graph. These descriptions are in line with the global versus the pointwise approach formulated by Even (1998). We will give two examples of strong vs. weak heuristics.

Someone having to graph $y = \ln(x-4)$ can recognize this function as a logarithmic function with a graph that increases and has a vertical asymptote. A strong heuristic is to sketch the standard function $y = \ln(x)$ and use a translation, because the focus is on the whole graph. A weak heuristic is to find the zero and vertical asymptote and calculate some points, because in that case the person only looks locally and tries to construct the graph with this local information.

When someone has to graph a formula and does not even partly recognize the graph, making a table is considered a weaker heuristic than qualitative reasoning about infinity behavior or about symmetry, because of the difference in local and global information about the graph.

Thus, regarding graphing formulas we have identified six levels of recognition (from direct recall to no recognition at all), and strong and weak heuristics. From the expertise literature, we have learned that recognition guides heuristic search. This interplay between recognition and heuristics results in a two-dimensional framework: For every level of recognition, we can formulate strong and weak heuristics, and so construct a two-dimensional framework (Table 2.1).

Table 2.1 Two-dimensional framework

		Heuristic search (strong → weak)				
(low ← high) Levels of recognition	A	A1. Graph is instantly recognized as a whole				
	B	B1. Recognition of family (with characteristics); possible graphs are known	B2. Search for ‘parameters’ of the graph	B3. Investigate the family characteristics, for instance via zeroes, derivative		
	C	C1. Split formula in sub-formulas, graphs of sub-formulas being known	C2. Compose the graphs by qualitative reasoning	C3. Compose the graphs by making a table		
	D	D1. Characteristic aspect of graph is recognized; rest of graph is unknown				
	E	E1. Graph is not recognized; algebraic formula is starting point for strategic exploration	E2. Qualitative reasoning for instance about domain, or vertical asymptote, or symmetry, or infinity behavior, or increase/decrease	E3. Algebraic manipulation	E4. Strategic search, for instance for zeroes or extreme values (via derivative)	E5. Calculate strategically chosen point(s)
	F	F1. No recognition at all	F2. Standard repertoire of research	F3. Make table with random x values		

2.2.7 Research questions

We wanted to check if the differences between the strategies for graphing formulas of experts, teachers, and learners could be accommodated within the framework of Table 2.1. We also wanted to check how often a person would use more than one strategy, so that the description of sequences of strategies may result in a “path” in the framework. At this stage of our research, having established the framework, we focused first on strategies of experts. Experts are expected to have more knowledge, and thus to recognize more and to use stronger heuristics. Therefore we expected their paths in the framework to contain fewer steps compared to the non-experts, and to be situated predominately on the upper side (more recognition), and/or on the left side (strong heuristics) of the framework in comparison with the paths of non-experts.

This raised the following questions:

1) Does the framework describe strategies in graphing formulas appropriately and discriminatively?
2) Which strategies do experts use in formula-graphing tasks?

2.3 Method

This study can be characterized as an exploratory study, allowing for a portrayal of the richness of the situation in which knowledge elements and heuristics have to be established. Before we started the experimental part of our research, we discussed formula-graphing strategies in interviews with three well-known researchers in mathematical education. These interviews provided us with an inside view of the levels of recognition and heuristics used in graphing formulas such as $y = x + 4/x$ and $y = \sqrt{1+x^2}$.

2.3.1 Tasks

To elicit the participants' strategies we developed two tasks. See Figure 2.3. Because we wanted to move participants out of their recognition zone, these tasks contained complex formulas and graphs. To challenge the participants' adaptive expertise, we presented them with a new situation, such as task B below, which required them to work from graph to formula, which is the reverse of the usual order. Such a reverse task calls on the same thinking processes, because participants are expected to graph formulas when they elaborate and test potential solutions.

During both tasks, participants were allowed to use only paper and pen. The available time was 10 minutes for each problem. Participants were asked to think aloud; this was not expected to disturb their thinking process and should give reliable information about their problem-solving activities (Ericsson, 2006).

Tasks

In the first task, we ask you to graph a formula. In the second task, we ask you to find a formula that fits a given graph. We are interested in your strategies. Please voice your thoughts and think aloud while solving these tasks. Time indication: 10 minutes per task.

Task A. Graph the formula $y = 2x\sqrt{8-x} - 2x$

Task B. Find a formula that fits the graph

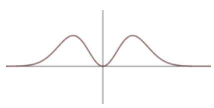


Figure 2.3 Task A and task B

2.3.2 Participants

To make sure we covered the full range of possible strategies, we invited highly regarded mathematical experts along with a range of teachers to participate. Five experts were selected, from different backgrounds: three mathematicians who teach first-year students at university, one author of a mathematical textbook, and one teacher educator. All of them had a master's degree or PhD in mathematics and had more than ten years of experience in teaching. In their education and in their work, they often have to graph formulas, and therefore we considered them experts in graphing formulas. We assigned the letters P, Q, R, S, and T to our five experts.

We would expect the experts' paths in the framework to be located predominantly on the upper and left sides of the table. In order to provide a contrast with the experts we invited three math teachers to solve the same tasks. Teachers are not novices, but we do not assume all teachers to have the same level of expertise in graphing formulas. We would expect teachers' paths to be situated more on the lower and right sides of the framework. The teachers, labeled U, V, and W had 30, 6 and 2 years of teaching experience, respectively.

2.3.3 Coding for task A

All participants' performances were videotaped and transcribed, and we used the framework to analyze the results. The transcriptions of the think-aloud protocols were cut into sections to allow encoding according to our framework. Unspoken actions and observations of the experimenter were indicated by [….]. The recognition levels mentioned in the Theory section were used to start the encoding. Coding was done according to the following instructions:

- If a participant immediately recognizes the graph $y = 2x$ as a straight line or $y = \sqrt{x}$ as a half horizontal parabola, encode A1 (the graph is instantly recognized)
- If a participant sees $y = \sqrt{8 - x}$ as a member of the family of square root functions and uses transformations on $y = \sqrt{x}$ encode B1-B2; however, if the participant does not use transformations but instead determines starting point (8,0) and calculates one or more points to determine whether the graph is to the right or to the left, encode B1-B3
- If a participant sees $y = 2x\sqrt{8 - x}$ and decomposes this formula into $y = 2x$ and $y = \sqrt{8 - x}$ (C1), graphs both formulas immediately (A1), and then multiplies the graphs by qualitative reasoning (C2), encode C1-A1-C2. If a participant multiplies these graphs by multiplying several y values, encode C1-A1-C3

- If a participant sees $y = 2x\sqrt{8-x}$ and describes the graph around (8,0) in terms of 'from (8,0) it starts to the left with a vertical tangent' (D1), and then makes a table with several well-chosen x values (E5), encode D1→E5
- If a participant sees $y = 2x\sqrt{8-x} - 2x$, starts factorizing and calculates zeroes, domain, and extreme values, encode E1-E3-E4-E2-E4

All fragments in the protocols were coded by two coders working independently. The few differences were discussed, and agreement was reached in all cases. The encodings were displayed in the framework. If the same encoding appeared several times in a row, this was noted as only one point in the relevant segment of the framework. For every participant this resulted in a path inside the framework. In addition, the time needed to solve the problem and the extent to which a participant was successful was indicated.

2.3.4 Task B: From graph to formula

From the two-dimensional framework, we derived a special framework to analyze the performances on task B. We used the same recognition levels; the heuristics on every recognition level in the two-dimensional framework were "translated" into heuristics for this new task. For this derived framework similar encoding instruction were formulated.

Table 2.2 Derived framework for task B: from graph to formula

		Heuristic search (strong→weak)		
(low ←high) Levels of recognition	A	A1. Formula is instantly recognized		
	B	B1. Recognition of family of formulas ('is something like…')	B2. Searching for 'parameters' of the formula (e.g., translation or via zeroes)	
	C	C1. Graph is decomposed into subgraphs	C2. Finding formulas for subgraphs and composing the formula	
	D	D1. 'Parts of the graph give parts of formula' (part of formula is recognized)	D2. Adjusting a formula to characteristics via qualitative reasoning	
	E	E1. Mentioning algebraic formulas ('can it be something like this?') without direct link to graph	E2. Checking a formula via qualitative reasoning	E3. Checking a formula by general methods (zeroes, extreme values, table)
	F	F1. No recognition at all		

2.4 Results

2.4.1 Results: graphing a formula

We first present the results of task A, graphing the formula $y = 2x\sqrt{8-x} - 2x$. For all participants we encoded the protocols based on the framework. These encodings can be tabulated as in the example below. We illustrate this process for expert *P* (Table 2.3); the results for all participants are given in Table 2.4.

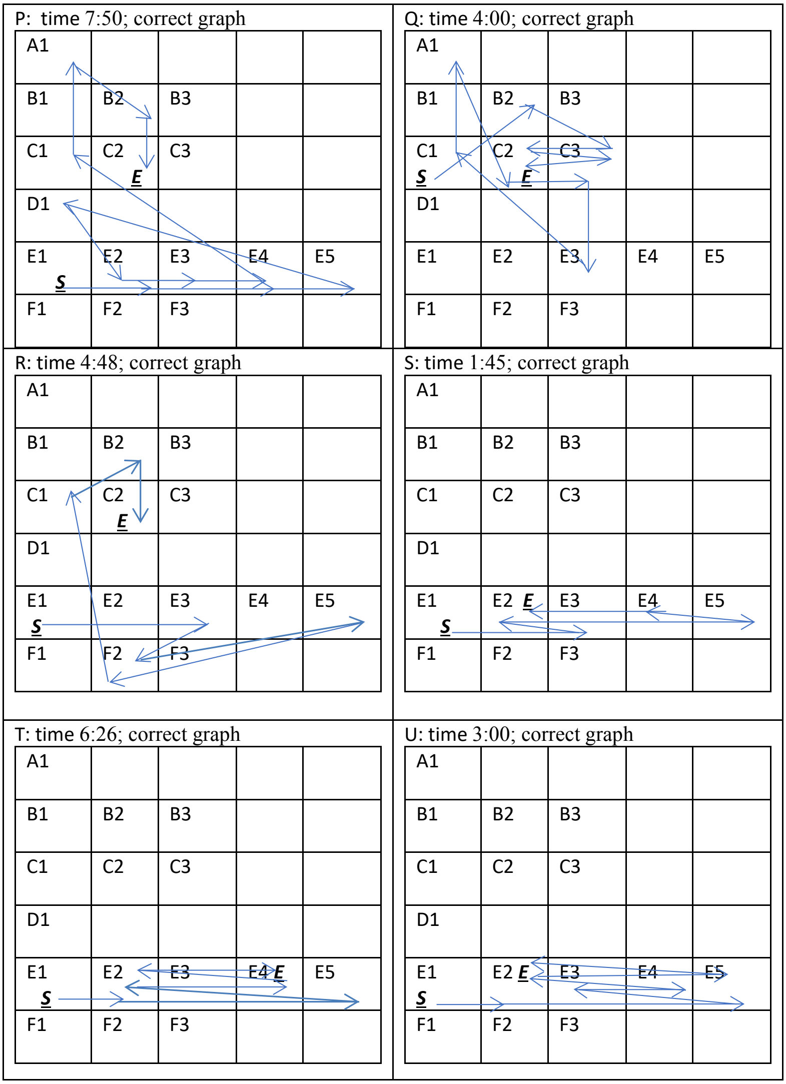
Table 2.3 Expert P's results on task A

Fragments	encoding
'yes, then I write down the formula; I always start like this' 'yes, I establish the domain: $x \leq 8'$	E1/E2
'anyway, the graph goes through (0,0)' [starts sketch]	E4
'if I take 8 then the graph is on -16' [draws point (8,-16)]	E5
'that will be vertical there' [draws vertical part of graph in (8,-16)]	D1
'what is happening if I look at minus infinity' [draws part of graph down left]	E2
[factorizes the formula and write $y = 2x(\sqrt{8-x} - 1)$]	E3
[calculates the derivative; makes a mistake with the chain rule; calculation gives an equation which has no solutions]	E4
'this is a hassle; there has to be a simpler way' 'I'll start once more; I have $\sqrt{8-x}$' [draws graph of $y = \sqrt{8-x}$]	C1/A1
'-1' [graph of $y = \sqrt{8-x} - 1$]	B2
'multiplies this by $2x'$	C2
'that means that the graph here is still a little lower' [points around $x = 8$]	C2
'and if I multiply by $2x$, anyway on this side it is still positive; and this will be negative' [points near $x = 8$]	C2
'and here it is 0' [points to the zero of of $y = \sqrt{8-x} - 1$]	C2
'the turning point should be here' [points near $x = 4$]	C2
'and here [points left of y axis] I multiply by something negative and the graph will go like this' [sketches graph left of y axis] 'and the graph will go down very quickly' 'it will result in this graph' 'and the zero will be at the moment that $\sqrt{8-x}$ equals 1; so $x = 7'$ [looks at the calculations of the derivative; but cannot find the mistake] 'well it has to be something like this; I made a mistake in the calculation of the derivative'	C2

Table 2.4 Results on task A

Experts															
P	E1/E2	E4	E5	D1	E2	E3	E4	C1/A1	B2	C2	C2	C2	C2	C2	C2
Q	C1/A1	C2	C3	E3	C1/B2	C3	C3	C2	C2	C2	C2	C3	C2	C2	
R	E1/E3	F2	F2	E5	F2	C1/B2	C2								
S	E1/E3	E2	E5	E4	E2										
T	E1/E2	E5	E5	E2	E4	E2	E4								
Teachers															
U	E1/E2	E5	E3	E4	E2	E5	E2	E2							
V	E1/E3	E4	E5	E5	E2	E4	E5	E2	E4	E4	E2	E5	F3		
W	E1/E2	E5	E5												

For every participant the data are displayed as a path in the framework. Starting point "***S***" and end point "***E***" are indicated, as well as the time (in minutes) needed, and whether the graph was correct, partially correct, or not correct (Figure 2.4).



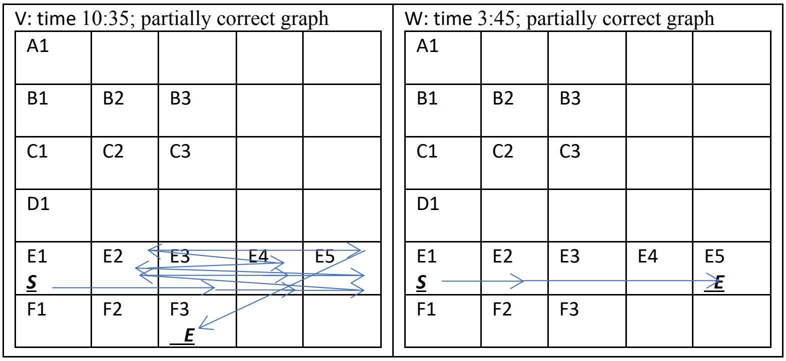


Figure 2.4 Results on task A in the two-dimensional framework

Figure 2.4 shows that in task A all experts found a correct graph, although T made a mistake when calculating the extreme value of the function. Experts solved this task in about five minutes on average. Four out of five experts started at recognition level E: the algebraic formula is the starting point for exploration. Only expert Q started by decomposing the formula into sub-formulas, and later considered a different decomposition.

Experts S and T started by exploring the algebraic formula and used qualitative reasoning. Both strategies can give fast results. Some experts (P and R) took an in-between position: they started by exploring the algebraic formula but later switched to decomposition. The protocols show their considerations, in which they state that they do not make sufficient progress and will start to look for alternative strategies: ‘this is a hassle, there has to be a simpler way’ (from the protocol of expert P). Only experts P, Q, and R used their repertoire of formulas that can be instantly visualized in this task.

All teachers started on recognition level E, which shows that they did not use their repertoire of formulas that can be instantly visualized. Two teachers were less successful in completing this task: their graphs were only partially correct. In general, the teachers’ paths are located in the lower and right sides of the framework.

2.4.2 Results on task B: from graph to formula

In order to analyze the performances on task B (finding a formula that fits a given graph), we encoded the fragments of the thinking-aloud protocols. These encodings are tabulated below. First, we illustrate this process for expert S (in Table 2.5); we give the results for all participants in Table 2.6.

Table 2.5 S's results on task B

Fragments	encoding
'You need an asymptote and at x=0 the graph is at zero; that [means] the graph is symmetrical, so I do something with x^2'	D1
'first I look for something that starts at 0 for positive values , increases and then will decrease to 0; it is hard' [pause] 'Around here is a maximum'	D2
'you can do something with x^2; $(x-1)^2$ has a minimum; the 1/….has a maximum '[writes down $1/(x-1)^2$]	
'but then it will go in this direction and at x=0 is not 0'	E2
'thus, if I multiply by x'	D2
[writes down $y = x/(x-1)^2$] 'then at x=0 it will be 0 and if x goes to infinity it will still go to 0'	
'This seems correct; I take x^2; so I get $y = x^2/(x^2-1)^2$	D2
'no that is not correct'	E2
´then instead of a minus I make + ; because otherwise I will get an asymptote [writes $y = x^2/(x^2+1)^2$]´	D2
´Let's see; again this is not correct; I want a maximum at x=1 not at x=0 ;	E2
If I look at $x/(x+1)$ or $x+1/x$? [pause] this is not correct´	
'Oh, this one might perhaps correct after all: $y = x^2/(x^2+1)^2$; seems OK; fine function'	E2

Table 2.6 Results for task B

Experts																
P	C1	C2	D2	E2												
Q	D1	C1	C2	E2												
R	B1	E2	D2	E2	C1/2	D2	E2									
S	D1	D2	E2	D2	D2	E2	D2	E2	E2							
T	B1	E2	D2	E2	D2	E2	C1/2	D2	E2	D2	D2	E2	E3	D2	D2	E2
Teachers																
U	D1	D2	E3	E2	D2	E2										
V	D1	D2	E2	D2	E2	D2										
W	D1	D2	E2	D2	E2											

In task B, four out of five experts found a correct formula: $y = x^2e^{-x^2}$ (experts P, Q, R) and $y = x^2/(x^2+1)^2$ (expert S). Halfway through, expert T (total time 6:26 minutes) hit on a correct formula ($y = x^2e^{-|x|}$) but did not recognize this as correct and proceeded without finding another correct formula. Teacher U found $y = x^2/(x^4+1)$ in 4:40 minutes. The other two teachers had more difficulties: V found $y = |x|/e^{|x|}$ in 6:20 minutes and W presented $y = (\frac{1}{2})^{x^2}$ within 1:15 minutes as a formula, which would fit the graph.

So, overall, most experts used encodings at level C, in which the graph is decomposed into subgraphs (expert R), or decomposed by describing subgraphs (experts P, Q, T). The strategies used most frequently, and used by every participant, were those of adjusting a formula to characteristics via qualitative reasoning, followed by checking a formula via qualitative reasoning.

2.4.3 Comparing choices of strategies

In both tasks, it was possible to use recognition strategies (levels A, B, C) or do an analysis of the algebraic formula (levels D, E, F). Figure 2.4 and Table 2.6 show whether a participant used a recognition strategy or not. In this way, strategies on the two tasks can be compared (Table 2.7).

Table 2.7 Comparison of strategies on tasks A and B

Expert	Task A strategy	time	Task B strategy	time
P	$\frac{1}{2}$ - $\frac{1}{2}$	7:50	1-0	3:43
Q	1-0	4:00	1-0	1:40
R	$\frac{1}{2}$ - $\frac{1}{2}$	4:48	$\frac{1}{2}$ - $\frac{1}{2}$	2:32
S	0-1	1:45	0-1	3:32
T	0-1	6:26 (P)	$\frac{1}{2}$ - $\frac{1}{2}$	6:26 (X)
U	0-1	3:00	0-1	2:36
V	0-1	10:35 (P)	0-1	6:22 (P)
W	0-1	3:45 (P)	0-1	1:14 (X)

1-0 indicates that the recognition strategy was used;
0-1 indicates that the recognition strategy was not used, only analysis of the algebraic formula;
$\frac{1}{2}$ - $\frac{1}{2}$ indicates that the participant started with an analysis of the algebraic formula and later switched to recognition strategies;
"X": the problem was not solved correctly; "P": the solution was only partially correct.

Most participants were consistent in their choice of strategy on both tasks; they used either recognition strategies or analysis of algebraic formulas (see Table 2.7). Both strategies can be successful and can give fast results.

The videotapes show that all participants started without any hesitations. They appear to recognize the type of task and do not seem to consider any general problem-solving heuristics.

2.5 Conclusions and discussion

Learners have difficulties reading algebraic formulas and the underlying process-object duality of functions. In graphing a formula, they have to read the formula and to use both a process and an object perspective. It was largely unknown what are effective and efficient strategies for graphing formulas. The purpose of our research was to identify a framework of strategies involving graphing formulas and to describe experts' strategies in graphing formulas.

2.5.1 Conclusions

The first aim of our research was to identify a framework with which we could describe formula-graphing strategies appropriately and discriminatively. We consider the framework *appropriate* if all strategies used by participants can be encoded in it. The results show that all statements from the protocols of all eight participants could actually be encoded within the two-dimensional framework. Therefore, we conclude that strategies used by the participants in our tasks can be described appropriately within the two-dimensional framework. The framework is *discriminative* if different strategies used by participants result in different paths in the framework. Figure 2.4 shows the differences and similarities in strategies, which also appear in the protocols. The videotapes and the protocols show that Q is more straightforward in his strategy choices, works faster and in a more straightforward way, and uses his domain knowledge and skills more efficiently. The different paths of experts P and Q in the framework, resulting from differences in strategy according to the protocols, are clearly seen: Q is faster (4:00 versus 7:50 minutes), Q's path contains fewer steps, and Q's steps are situated more at the upper side of the framework (indicating more recognition), and more on the left side (indicating stronger heuristics). Therefore, we conclude that the framework is also discriminative.

The second aim of our research was to describe experts' strategies in graphing-formula tasks. The experts in our research used a range of strategies in graphing formulas. Qualitative reasoning and recognizing and using formulas that can be instantly visualized by a graph seem to be the main strategies used by the experts in our research. Table 2.3 and Table 2.7 show that some experts focused on recognition and used their repertoire of formulas that can be instantly visualized by a graph. Other experts focused on analysis of algebraic formulas and used their strong heuristics, such as qualitative reasoning about characteristics as, for instance, domain, infinity behavior, and symmetry.

Expertise in graphing formulas does not involve calculations of derivatives. All our experts seemed to hesitate to start such calculations. In addition, when two of them did, they made mistakes.

We formulated task B in order to establish whether expertise can be used functionally. In this task, participants were forced to use their repertoire of formulas that can be instantly visualized. Results from Table 2.6 show that recognition was used to formulate hypotheses about the formula, and that formula graphing was used to test these hypotheses (E2, E3 in the derived framework). This shows that for task B the same thinking processes were needed as for graphing formulas. Since all fragments from the protocols could actually be encoded within the derived framework, the results for task B are in accordance with the two-dimensional framework.

Although in graphing formulas (task A) some of our experts did not use their repertoire of formulas that can be instantly visualized by a graph, when they were forced to, as they were by task B, most experts showed that they do have a large repertoire and were able to use that repertoire.

Not all our teachers solved these tasks adequately. As expected, teachers' paths are situated on the lower side of the framework, because they did not use high levels of recognition (levels A, B, or C). The variation in the teachers' performances (correct graph/formula, time needed) is large. The performance of the most experienced teacher (U) closely resembles that of expert S. Teacher U can be considered an expert in this domain of graphing formulas. Teacher V produced only partially correct solutions and needed many steps in his solving process, as well as a lot of time. Teacher W worked very fast, used weak heuristics, and produced inaccurate solutions.

2.5.2 Discussion

In literature, several aspects are mentioned which are important for graphing formulas (see Theory section) but there was still no framework to describe strategies involving graphing formulas. In our framework knowledge about expertise, about recognition, and about heuristic search for graphing formulas is integrated. From expertise literature, it is found that recognition and heuristic search are two components of expertise. To investigate these components, we used theory about reading formulas and the process and object perspectives.

For recognition a repertoire of basic functions (Eisenberg & Dreyfus, 1994), symbol sense (Arcavi, 1994), decomposition of algebraic expressions into smaller expressions (Ernest, 1990), and the classification of function-families are considered important aspects. Different levels of awareness have been formulated by Mason (2003). In this research, these aspects have been combined into a scale of recognition, from complete recognition to no recognition at all.

For heuristic search, we use aspects of process and object perspectives. The process perspective in which a function is seen as a calculation rule in which x and y values are linked, gives often only local information about the graph and is called a pointwise approach by Even (1993, whereas the global approach (Even, 1993) or covariational reasoning (Carlson, 2002; Oehrtman et al., 2008) gives information about a function's behavior on an interval or in a global way. We consider heuristics that result in information about intervals as strong and heuristics that result in local information as weak. In this way, heuristics are ordered into a scale of heuristics, form strong to weak.

We found that the framework did cover the strategies used by the participants. However, not all strategies in the framework were found in the participants' protocols (see Figure 2.4). An explanation might be the limited number of experts and teachers that could be included in the study. This limitation is due to the labor-intensive method for strategy assessment. Further research, with a larger group of experts and teachers, may provide more information about the strategies used in graphing formulas. In addition, students should be included so that the lower range of the framework, i.e., levels E and F, may be explored further.

Another aspect that could have influenced the strategies we found can be the choice of the formula and the graph used in tasks A and B. For instance, recognition level D ("characteristic aspect of graph is recognized; rest of graph is unknown") was found only once. In the earlier pilot interviews this level was used by experts in the case of rational functions such as $y = x + 4/x$ and $y = 4/(x^2 - 4)$. Future research, involving other functions, such as these rational functions, can provide information on whether alternative strategies not mentioned in the framework are used regularly.

Reflecting on the results of task A, we were surprised by the form of the paths in the framework (Figure 2.4). Although we expected the experts' paths in the framework to be located predominantly in the upper and left range of the framework, the paths of the experts S and T were situated in the lower range of the framework and the paths of experts P and R started in the lower range of the framework. Task A seems to have triggered our experts' strong heuristics and not their repertoire of formulas that can be instantly visualized by a graph. When a task has been practiced many times, a program of self-instruction can be developed. The familiar task triggers a set of personal metacognitive instructions, evoking specific activities (Veenman, Van Hout-Wolters, & Afflerbach, 2006). Although this can help in problem solving, routine can also be a risk. For chess, Saariluoma (1992) found that strong players tend to choose stereotyped solutions and sometimes miss non-typical, shorter solutions. Perhaps this can explain why some of our experts did not use their repertoire of formulas that can be instantly visualized by a graph.

In this research, a framework with hierarchies of recognition and heuristic search in graphing formulas has been defined. This framework can be used to assess expertise in graphing formulas. In this way, teachers' and students' current strategies can be compared with expert strategies. In addition, the framework might be used to indicate a development trajectory for teaching efficient strategies in graphing formulas. Further research is necessary to elucidate if, and how, the framework can indeed be helpful in learning and teaching to graph formulas.

CHAPTER 3

Graphing formulas:
Unraveling experts' recognition processes

This chapter is based on: Kop, P. M., Janssen, F. J., Drijvers, P. H., & van Driel, J. H. (2017). Graphing formulas: Unraveling experts' recognition processes. *The Journal of Mathematical Behavior*, *45*, 167–182.

Abstract

An instantly graphable formula (IGF) is a formula that a person can instantly visualize using a graph. These IGFs are personal and serve as building blocks for graphing formulas by hand. The questions addressed in this paper are what experts' repertoires of IGFs are and what experts attend to while recognizing these formulas. Three tasks were designed and administered to five experts. The data analysis, which was based on Barsalou and Schwarz and Hershkowitz, showed that experts' repertoires of IGFs could be described using function families that reflect the basic functions in secondary school curricula and revealed that experts' recognition could be described in terms of prototype, attribute, and part-whole reasoning. We give suggestions for teaching graphing formulas to students.

3.1 Introduction

Algebraic concepts, like functions, can be explored more deeply through linking different representations (Duval, 2006; Heid et al., 2013). Graphs and algebraic formulas are important representations of functions. Graphs seem to be more accessible than formulas (Leinhardt et al., 1990; Moschkovich et al., 1993). In addition, graphs give more direct information on covariation, that is, how the dependent variable changes as a result of changes of the independent variable (Carlson et al., 2002). A graph shows features such as symmetry, intervals of increase or decrease, turning points, and infinity behavior. In this way, it visualizes the "story" that an algebraic formula tells. Therefore, graphs are important in learning algebra, in particular in learning to read algebraic formulas (Eisenberg & Dreyfus, 1994; Kieran, 2006; Kilpatrick & Izsak, 2008; NCTM, 2000; Sfard & Linchevski, 1994).

Students have difficulties in seeing a function both as an input-output machine and as an object (Ayalon et al., 2015; Gray & Tall, 1994; Oehrtman et al., 2008; Sfard, 1991). Graphs appeal to a gestalt-producing ability, and in this way can help to consolidate the functional relationship into a graphical entity (Kieran, 2006; Moschkovich et al., 1993). Graphs are also considered important in problem solving. Graphs are used for understanding the problem situation, recording information, exploring, and monitoring and evaluating results (Polya, 1945; Stylianou & Silver, 2004).

So, the ability to switch between representations, representation versatility, in particular conversions from algebraic formulas to graphs, is important in understanding algebra and in problem solving (Duval, 2006; NCTM, 2000; Stylianou, 2011; Thomas et al., 2010).

In a previous study a framework was developed to describe strategies for graphing formulas without using technology (Kop et al., 2015). In the framework, it is indicated how recognition guides heuristic search. When one has to graph a formula there are different possible levels of recognition: from complete recognition (one immediately knows the graph) to no recognition at all (one does not know anything about the graph). For every level of recognition the framework provides strong to weak heuristics.

For the two highest levels of recognition the graph is completely recognized or the formula is recognized as a member of a function family whose graph characteristics are known. For instance, at the highest level of recognition the graph of $y = x^2$ is instantly recognized as a parabola with minimum (0,0). At the second level of recognition, $y = 4 \cdot 0.75^x + 3$ is recognized as a member of the family of decreasing exponential functions, and so the horizontal asymptote is read from the formula. In this way the graph can be instantly visualized. Another example at this level: $y = -x^4 + 6x^2$ is recognized as a polynomial function of degree 4; because of the negative head coefficient its graph has an M-shape or an Λ-shape; a short investigation of, for instance, the zeroes will instantly give the graph.

At these two highest levels of recognition in the framework, formulas can be instantly linked to graphs. Therefore, these formulas are defined as *instantly graphable formulas* (IGF). A large set of IGFs is beneficial to proficiency in graphing formulas. The current study was focused on experts' recognition processes when dealing with IGFs. For this study we defined an expert as a person with at least a master's degree in mathematics and at least 10 years of experience teaching at the secondary or college level, with experience in graphing formulas by hand. Although these experts are expected to be able to instantly link many formulas to graphs, their repertoires of IGFs remain unknown. In addition, we investigated what experts attend to when recognizing IGFs. This information might give suggestions for a repertoire of IGFs for students and for a focus in teaching students IGFs.

3.2 Theory

3.2.1 Cognitive units as building blocks

IGFs can be seen as building blocks in thinking and reasoning with and about formulas and graphs. Barnard and Tall (1997) introduced the concept of "cognitive unit", an element of cognitive knowledge that can be the focus of attention altogether at one time. For experts, well-connected cognitive units can be compressed into a new single cognitive unit which can be used as just one step in a thinking process (Crowley & Tall, 1999). In this way experts' knowledge is well organized in hierarchical mental networks with complex cognitive units, which can be enlisted when necessary (Campitelli & Gobet, 2010; Chi et al., 1981; Chi, 2011).

As IGFs are cognitive units in graphing formulas, they can be combined (addition, multiplication, chaining, etc.) and can form new, more complex IGFs. For instance, when dealing with $y = -x^4 + 6x^2$, novices may recognize the IGFs $y = -x^4$ and $y = 6x^2$ and have to combine these two IGFs to draw a graph, whereas $y = -x^4 + 6x^2$ is an IGF for experts, who recognize a fourth degree polynomial function. For experts, a formula like $y = x^2 - 6x + 5$ can trigger other cognitive units, like "its graph is a parabola with a minimum value", and the equivalent formulas $y = (x-1)(x-5)$ and $y = (x-3)^2 - 4$, which can give information about the zeroes and the minimum value, etc. Experts are expected to have more, and more complex, IGFs than novices, which generally enable them to graph formulas with fewer demands on the working memory (Sweller, 1994).

The current study was focused on recognition: in particular, which formulas and/or function families were instantly recognized by experts and how the recognition processes can be described.

3.2.2. Recognition described using Barsalou's model with prototype, attribute, and part-whole reasoning

Barsalou (1992) showed how human knowledge is organized in categories or concepts. People construct these categories based on attributes. When a task requires a distinction to be drawn between exemplars of a category, people construct new attributes and in this way new categories (Barsalou, 1992). For instance, for the concept bird, attributes (variables) like size, color, and beak, with several values, can be used to distinguish different exemplars. Categories can have a large diversity of exemplars, but have a graded structure (Eysenck & Keane, 2000; Barsalou, 2008). Some exemplars in a category are more central to that category than others; these are called prototypes. For instance, a robin is considered a more typical example of a bird than, for instance, a chicken or a penguin. When dealing with exemplars of a category, people tend to associate prototypical features with these exemplars (Barsalou, 2008; Schwarz & Hershkowitz, 1999). The tendency to reason from prototypes can pose problems. Since concept formation is not necessarily done using pure definitions, Watson and Mason (2005) emphasized the need to go beyond prototypes and to search for the boundaries of a concept. In this way one becomes aware of the dimensions of possible variation and in each dimension of the range of permissible change (Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006; Sandefur, Mason, Stylianides, Watson, 2013; Watson & Mason, 2005) . The personal example space, the collection of examples and the interconnection between the examples a person has at their disposal (the accessible example space), play a major role in how a person makes sense of the tasks he/she is confronted with (Watson & Mason, 2005; Goldenberg & Mason, 2008). Vinner and Dreyfus (1989) used concept image to emphasize the personal character of people's mental networks. These concept images determine what a person “sees” when dealing with concepts or categories, and are used in rapid identification.

Schwarz and Hershkowitz (1999) used prototypicality, attribute understanding, and part-whole reasoning as aspects to portray students’ concept images of functions. We discuss these three aspects below.

Prototypicality refers to the prototypes (prototypical exemplars) a person knows and uses. Prototypes can be defined as the exemplar(s) with the set of highest frequency of attribute values in the category or with the highest correlation with other exemplars in the category (Barsalou, 1992). Prototypes are the examples that are acquired first and are usually the examples that have the longest list of attributes: the critical attributes of the category and the self-attributes (non-critical attributes) of the exemplar (Schwarz & Hershkowitz, 1999). Prototypes are used as a reference point for judging membership of the category: an exemplar is judged to be a member of a category if there is a good match between its attributes and those of the category prototype (Barsalou, 2008; Eysenck & Keane, 2000). When asked for a prototype of a category, it is expected that a person will not use a definition of prototype but will use a general idea about what prototypes are: namely, the most central exemplar(s) of a category from their personal perspective. As a consequence, when dealing with a category, the prototypes are the first examples that come to one’s mind and are the natural examples that are used without any explanation. Examples in the domain of graphing formulas include prototypical formulas like $y = x^2$ and $y = x^3$, with their prototypical graphs. In this study we used the term prototype reasoning in this way.

Attribute understanding can be defined as the ability to recognize the attributes of a function across representations (Schwarz & Hershkowitz, 1999). For instance, from the formula $y = (x - 1)(x - 5)$, it is concluded that its graph is a parabola, it has zeroes at $x = 1$ and at $x = 5$ and a symmetry axis at $x = 3$. These attributes or properties of this function can be recognized in the graphical, tabular, and algebraic representations.

In his property-oriented view of functions, Slavit (1997) used properties (or attributes) like symmetry, monotonicity, horizontal and slant asymptotes, intercepts (zeroes), extrema, and points of inflection.

Depending on the task, people construct attributes to be able to distinguish exemplars: in this study, formulas and graphs (Barsalou, 1992). To distinguish different graphs of fourth degree polynomial functions in Figure 3.1, one can use attributes like symmetry, infinity behavior, number of turning points, number of zeroes, and location of zeroes relative to the *y*-axis. When relating formulas and graphs, as in graphing formulas, one chooses or creates attributes to focus on features of formulas and graphs. We call this reasoning about attributes and their values attribute reasoning.

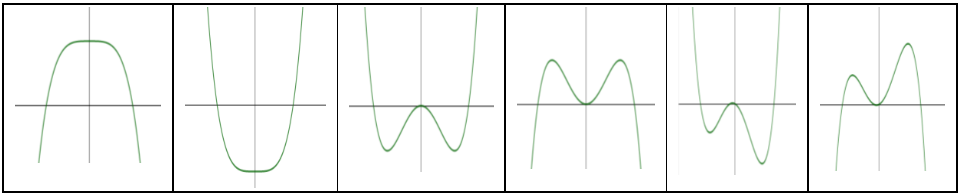


Figure 3.1 Graphs of fourth degree polynomial functions

Part-whole reasoning refers to the ability to recognize that different formulas or different graphs relate to the same entity: in this case, to the same function. In the graphical representation, different scaling can result in different pictures of graphs belonging to the same function. In the algebraic representation, formula manipulation can result in different formulas of the same function: for instance, $y = x^2 - 4x$, $y = (x-2)^2 - 4$, and $y = x(x-4)$. From these different formulas different attributes of the graph can be read. Therefore, part-whole reasoning is important in the recognition of IGFs.

For attribute reasoning and part-whole reasoning one has to grasp the structure of a formula. In the literature this is called symbol sense (Arcavi, 1994). Symbol sense is a very general notion of "when and how" to use symbols and has several aspects, such as the ability to read through algebraic expressions, to see the expression as a whole rather than a concatenation of letters, and to recognize its global characteristics (Arcavi, 1994). Pierce and Stacey (2004) used algebraic insight to capture the symbol sense in transformational activities in the "solving" phase of problem solving (Pierce & Stacey, 2004). The algebraic insight is divided in two parts: algebraic expectation and the ability to link representations. Algebraic expectation has to do with recognition and identification of objects, forms, key features, dominant terms, and meanings of symbols (Kenney, 2008; Pierce & Stacey, 2004). Algebraic insight is shown when a person has expectations about graphs that are linked to features of the symbolic representation and when equivalent algebraic expressions are recognized (Ball, Stacey & Pierce, 2003; Pierce & Stacey, 2004).

The three aspects prototype, attribute, and part-whole reasoning from Schwarz and Hershkowitz can be used to describe the recognition process in graphing formulas. A

Barsalou model for recognizing IGFs is formulated in Figure 3.2. In the case of graphing formulas, it is difficult to mention all possible values. For instance, the attribute "zeroes" can have values like 0,1,2,3, etc. to indicate the number of zeroes, but also the location can be used as values of an attribute (for instance, a zero at $x = 5$). For the sake of readability, the values belonging to the attributes are omitted in Figure 3.2.

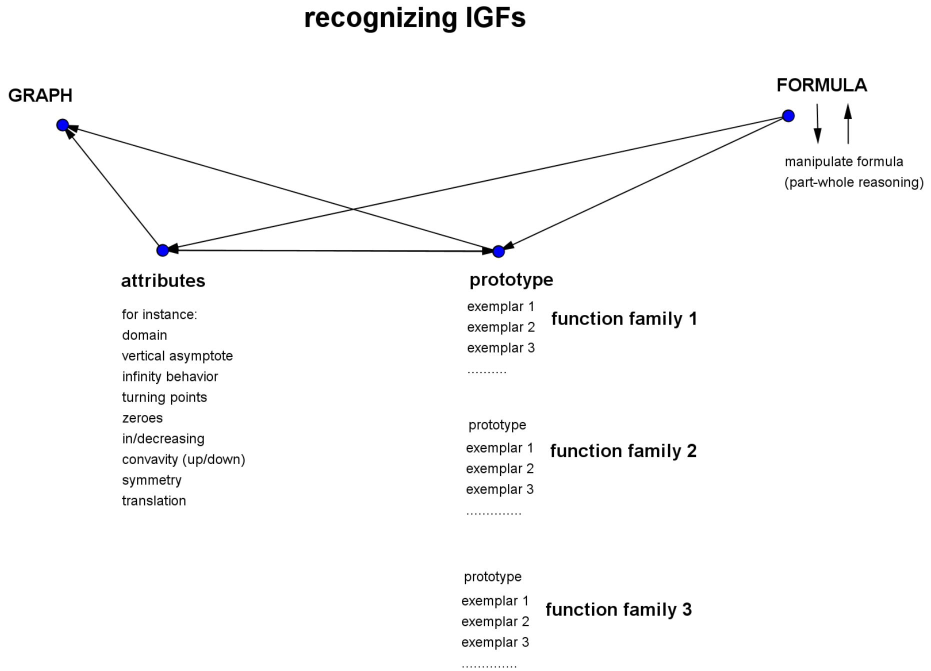


Figure 3.2 IGFs in the form of a Barsalou model based on Schwarz and Hershkowitz (1999)

The Barsalou model in Figure 3.2 shows how function families are constructed by using value sets on a set of attributes and allows a detailed description of how formulas can be linked to graphs, and so of the recognition of IGFs. Starting with a formula (on the right side of Figure 3.2), there are several possibilities: the formula can be manipulated (part-whole reasoning) into another formula, the formula can be recognized as a member of a function family, or the formula can be recognized as a prototype of a function family. It is then possible that the graph is directly known, or that, using attribute reasoning, a graph can be visualized.

Some examples can illustrate this recognition process. In IGF $y = 4 \cdot 3^x + 2$ the prototype 3^x can be recognized (prototype reasoning), and via a translation (attribute reasoning) the graph can be visualized. In IGF $y = -2x(x-3)(x-6)$, the prototype x^3 can be recognized, $-x^3$ as a reversion (attribute reasoning), and via zeroes at $x = 0$, $x = 3$, $x = 6$ (attribute reasoning) the graph can be visualized. However, when $y = -2x(x-3)(x-6)$ is not recognized as a member of a function family or prototype of a function family, the formula is not an IGF (Kop et al., 2015). In this case the graph has to be constructed by, for instance, reasoning about attributes like infinity behavior and zeroes. If, when graphing $y = 4x^{-2}$, the formula can be rewritten to $y = 4/x^2$ (part-whole reasoning) and recognized as a $1/x^2$ (prototype reasoning), the formula is an IGF. But when from the formula $y = 4/x^2$ it is read that it has a vertical asymptote at $x = 0$, and that all outcomes are positive and when $x \to \pm\infty$ then $y \to 0$ (infinity behavior) , then we say that the graph is constructed through qualitative reasoning (Kop et al., 2015), and so the formula is not an IGF.

3.2.3 Global and local perspectives

Covariational reasoning is essential for graphing formulas. In covariational reasoning, one is able to imagine running through all input-output pairs simultaneously and so to reason about how a function is acting on an entire interval of input values (Carlson et al., 2002). In recognizing IGFs one has to have a picture of the function as an entity. In the literature this perspective of the function, seeing the function as a whole, is also addressed as the object or global perspective (Confrey & Smith, 1995; Even, 1998; Gray and Tall, 1994; Oehrtman et al., 2008; Sfard, 1991). There is also another perspective of the function, namely, to see a function as an input-output machine. This perspective has to do with the fundamental view on functions (what it means that a certain y-value belongs to a given x-value), and is addressed as the pointwise, process, or correspondence perspective. Switching between both kinds of perspective is necessary for reasoning about functions. Slavit (1997) spoke about the local and global nature of functional growth properties in addressing both kinds of perspective (Slavit, 1997). The global growth properties concern attributes like symmetry, monotonicity, horizontal and slant asymptotes, integrability, and invertibility, whereas the local properties are about extrema, intercepts, cusps, and points of inflection. In an in-between class, Slavit also mentioned continuity, sign, differentiability, domain, and range. Graphs can be described using these properties or attributes. Before the current research, it was unknown which attributes experts use in recognizing IGFs.

3.2.4 Research questions

In the current study we focused on experts' repertoires of formulas that can be instantly visualized using a graph (IGFs) and on their concept images of IGFs, with attributes, prototypes, and part-whole reasoning. We expected that experts would have large repertoires of IGFs that are structured in categories. However, we did not yet know what an expert repertoire of IGFs would be.

We expected experts to be able to manipulate algebraic formulas (part-whole reasoning), to use symbol sense and in particular algebraic insight, and to use sets of attributes with value sets to distinguish different graphs. However, we did not know which prototype, attribute, and part-whole reasoning they would use in linking formulas and graphs of IGFs.

This leads to the following research questions:

Can we describe experts' repertoires of instant graphable formulas (IGFs) using categories of function families?

What do experts attend to when linking formulas and graphs of IGFs, described in terms of prototype, attribute, and part-whole reasoning?

3.3 Method

The current study can be characterized as an exploratory study, in which we investigated "snapshots" of experts' concept images of function families with their algebraic formulas and graphs.

3.3.1 Tasks

Three different tasks were developed to elicit the experts' repertoires of IGFs and to explore the experts' prototype, attribute, and part-whole reasoning: a card-sorting task, a matching task, and a multiple-choice task.

Card-sorting tasks are often used in eliciting structured knowledge (Chi et al., 1981; Jonassen, Beissner, & Yacci, 1993; De Jong & Ferguson-Hessler, 1986; Goldenberg & Mason, 2008; Sandefur et al., 2013).

In task 1, 60 formulas were given, and the participants were asked to categorize them according to their graph. After this, they were asked to give a name and a prototypical formula for each of their categories. We structured this task by adding graphs to the cards showing the formulas. When such tasks are given without structuring beforehand, getting a complete picture or comparing the results can pose problems, because of the different criteria that can be used to sort the cards (Ruiz-Primo, 1996). Because we add four graphs to the 60 cards with formulas, the participants were explicitly compelled to focus on the graphs of the formulas. We did not indicate whether a participant should discriminate between parabolas with a maximum or minimum because the level of detail can be an indicator of expertise. Figure 3.3 shows 20 cards from task 1. Most of these formulas, but not all, are related to one of the basic function families, which are studied in grades 10-12: $y = x^n,\ y = a^x$, $y = \log_2(x)$, $y = 1/x$, $y = \sqrt{x}$, $y = \ln(x)$, $y = \mathrm{e}^x$. Since we used the basic functions from secondary school curricula, we expected that many formulas, but not all, would be IGFs for the experts. This categorization task gave information about dimensions of variation and the range of permissible change experts used in discriminating graphs. The names given for the different categories with the prototypes gave insight into the graph families and thus in the attribute and value sets experts used.

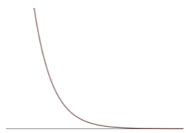
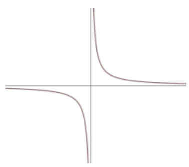
10^{-2x+5}	$1-5/(x+1)$	$2x^3(6-x)$	$6-2^x$
x^4-16x^2+28	$2\sqrt[3]{x}$	$3x^{-2}$	$4\sqrt{10-x}$
$(x^2-7)^2$	$(x+3)^4-9$	$(2x^{\frac{1}{3}})^5$	$(1-x)(2+x)+x^2$
$2x\sqrt{x}$	$(100x)^{\frac{1}{2}}$	$x-4/x$	$\sqrt{8-x^2}$
			

Figure 3.3 A number of the cards used in task 1

In Task 2, the matching task, a list of 40 formulas was given and the participants were asked to select the correct alternative out of 21 alternatives: 20 graphs and one alternative stating "none of these". This last alternative was provided to discourage guessing. In this task the focus was on instant linking of formulas to the global shape of graphs. Therefore, a strict time limit was used to encourage recognition and to discourage construction of a graph. We chose a matching task with many alternatives rather than a graphing task to indicate the level of detail that was needed: the experts had to recognize the global shape of the graph of the given formula.

The formulas used in this task resembled the formulas used in the first task. The following are some examples: $y = 2x(x-2)(x+4)$; $y = 6x^2 - 2x^4$; $y = e^{2x} + 1$; $y = x - 4/x$; $y = 4 - 2x + x/4$; $y = 4/2^x$; $y = \sqrt{x-6} + 2$; $y = 2x^{-4}$; $y = 2(x-1)^4 - 4$; $y = \sqrt{8-x^2}$; $y = \ln(4/x)$; $y = 9x/\sqrt[3]{x}$; $y = \ln(e^2 \cdot x)$. Eight of the alternative graphs are shown in Figure 3.4.

Task 2 was also developed to elicit participants' repertoires of IGFs. Therefore some functions were added that do not belong to the function families of basic functions, for instance, $y = \sqrt{8-x^2}$, $y = 30/(x^2 - 16)$, $y = x - 4/x$, because we wanted to investigate the boundaries of the experts' repertoires of IGFs. Because the formulas used were similar to those in task 1, this task was used to validate the results of task 1. When, for instance, in task 1 no distinction was made between increasing and decreasing parabola, but in task 2 this distinction was made, it was concluded that the participant could indeed make such a distinction.

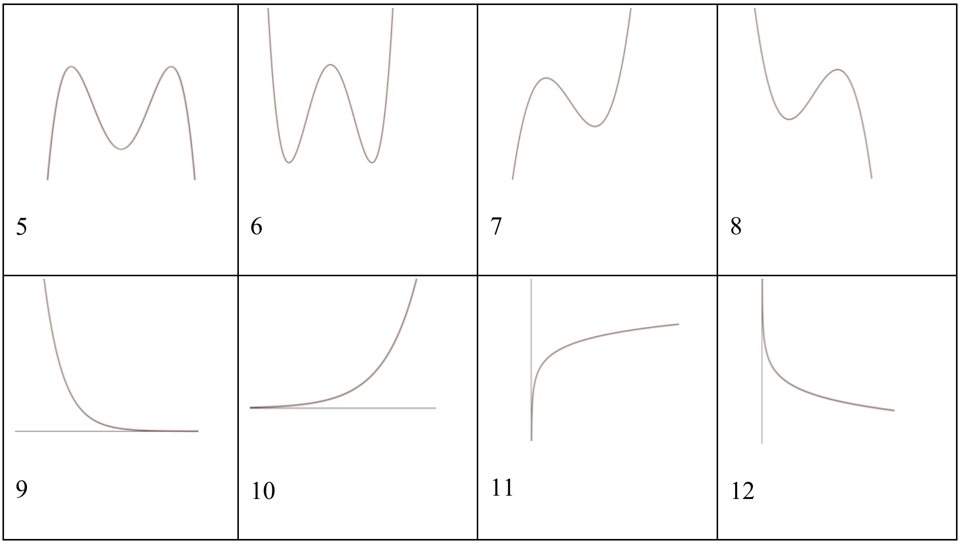
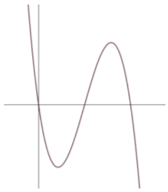


Figure 3.4 Some alternatives of task 2

Tasks 3A and 3B, thinking aloud multiple-choice tasks, were developed to elicit the participants' prototype, attribute, and part-whole reasoning and in this way to get more detailed knowledge of the participants' concept images. The participants were asked to choose the correct alternative out of four alternatives. A similar task was used by Schwarz and Hershkovitz (1999) in their study of concept images of functions. Both tasks consisted of six items. In task 3B a formula was given and the experts had to find the correct graph. In task 3A a graph was given and the experts had to provide a formula. In general, tasks like 3A are considered to be more challenging. But this is not clear when dealing with the function families of well-known basic functions. In this way we got more detailed information about the experts' concept images of IGFs. Three examples of this task are shown in Figure 3.5.

Task 3A-4



Which formula(s) can fit this graph:

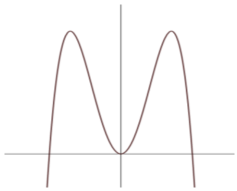
A. $y = x(x-3)(x-6)$

B. $y = -2x(x-2)(x-4)$

C. $y = x(3-x)(x-6)$

D. $y = -2x(x+3)(x+6)$

Task 3A-6



Which formula(s) can fit this graph:

A. $y = -x^4 + 9$

B. $y = -x^4 + 9x^2$

C. $y = -x^4 - 9x^2$

D. $y = -x^4 + 9x^3$

Task 3B-3: Indicate which graph(s) can fit $y = 100 - 50 \cdot 0.75^x$

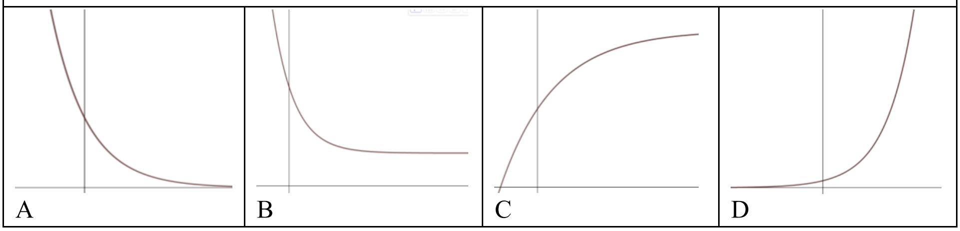


Figure 3.5 Some examples of task 3: task 3A-4, 3A-6, 3B-3

The formulas were again chosen from the same set of functions as in tasks 1 and 2. Participants had to consider all alternatives because more than one alternative could be correct.

In tasks 1 and 2 the focus was on sketches of graphs; in this task, more detailed answers were needed. For instance, in tasks 1 and 2 it was not necessary to distinguish $y = -2x(x-2)(x-4)$ and $y = -2x(x+3)(x+6)$, but in task 3 this distinction had to be made (see task 3A-4 in Figure 3.5).

3.3.2 Participants

Five mathematical experts were invited to participate in this study. We assigned the letters P, Q, R, S, and T to our five experts. The experts had different backgrounds: two mathematicians who had been teaching calculus and analysis to first-year students at university (Q, R), one author of a mathematics textbook series, who had been a teacher in secondary school (T), one math teacher who was involved in the National Math Exams and had been a secondary school teacher (S), and one math teacher educator in university (P). All had a master's degree in mathematics and two had a PhD in mathematics (Q, R). All of them had been working as a teacher at university or in secondary education for more than 20 years and had been graphing many formulas without technology during their education and during their whole teaching career. Therefore, we considered them experts in graphing formulas.

3.3.3 Data collection procedure

Written instructions were handed out for every task, together with an indication of the time needed to perform the task. For task 1, a time indication of maximum 40 minutes was given; for tasks 2 and 3, 20 minutes. For all tasks, the time needed was recorded, as the time required to perform a task can be an indication of expertise. During the tasks the first author only emphasized the need to keep on thinking aloud when the experts stopped talking. After each task, the first author asked the experts to look back and to describe the strategies they had used in the task. The interviews were videotaped.

In task 1, the card-sorting task, 60 cards were laid on a table and the participants could physically group the formulas into different categories. Afterwards, the categories were glued on a large sheet. The participants then wrote the category names and the prototypical formulas for each category. In task 2 and task 3, the participants filled in the answers on a form.

During tasks 1 and 3 the participants were asked to think aloud; this was videotaped. Thinking aloud is considered to give reliable information about the problem-solving activities without disturbing the thinking process (Ericsson, 2006). For task 3 the thinking-aloud protocols were transcribed in order to analyze the prototype, attribute, and part-whole reasoning.

3.3.4 Data analysis

Task 1: The aim of task 1 was to gather information on which categories experts use in their repertoires of IGFs. It was expected that experts would use salient, global properties of graphs, like symmetry, in/decreasing, vertical asymptotes, infinity behavior, and number of turning points, to categorize their IGFs. Based on these salient properties, the first author made a theoretical, hypothetical experts' categorization before the start of this study. The categorizations of the five experts were compared with each other and with the first author's categorization. Based on these findings a common categorization was constructed. This was done in several steps. First, common elements in the categories and prototypes in the experts' categorizations and the first author's categorization were determined. From these findings a preliminary common expert categorization was formulated. In the second step, the level of detail was considered. A higher level of detail meant that subcategories were used. If one or more experts used a higher level of detail, then this level of detail was used in the (final) common expert categorization. In the last step, the distances between individual categorizations and the common expert categorization were calculated. We considered whether small adjustments in the common expert categorization would result in a lower minimum of the total of all distances. When no progression could be made, the final common expert categorization was found.

To determine the distance between an individual categorization and a common categorization, the following protocol was used:

- If the individual categorization had the "same" category but a formula was not mentioned or did not belong to that category, then the distance increased by +1
- If no subcategories were made in the individual categorization and the common expert categorization made a distinction between increasing and decreasing, then the distance increased by +2 (for instance, no subcategories between parabolas with maximum and parabolas with minimum gave an increase of the distance by +2 if the common expert categorization made this distinction)
- If two categories of the common expert categorization were merged in the individual categorization (other than the distinction between increasing and decreasing), then the distance increased by +4 (for instance, 3rd and 4th degree functions were put together in one category)
- If a completely new category, different from the common expert categorization, was formulated, then the distance increased by +6.

Task 2

In this task the numbers of mistakes per expert were counted. The mistakes were indicated in a table in order to see whether they were made in particular function families.

Task 3

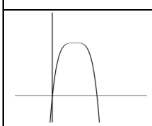
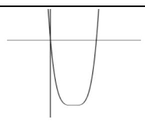
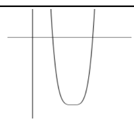
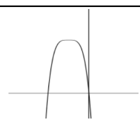
To analyze the results of task 3, the transcripts were cut into fragments which contained crucial steps of explanations: idea units. Idea units are primitive elements in the justifications of participants (Schwarz & Hershkowitz, 1999). These idea units were encoded using the elements from Figure 3.2: prototype, attribute, or part-whole reasoning.

Since prototypes are the natural examples of categories that can be used without any explanation, graphs and formulas that a participant used as the start of a reasoning process were considered prototypes for the expert. If no prototype reasoning was used or function family was mentioned, we said that the formula was not an IGF, and that the graph was constructed.

The fragments of the protocols were encoded as follows:

- pr (prototype reasoning) : only a prototypical exemplar was mentioned; for instance, “it looks like a log”, “it is an x in the power 6”, “it is an expo”, “it is an oscillation”. If a function family was mentioned, like in “it is an exponential function” or “fourth degree polynomial”, this was considered prototype reasoning
- att (attribute reasoning): an attribute was mentioned; for instance, “this one has a vertical asymptote at $x=0$”, “it is always positive”, “it goes to minus infinity”.
- pw (part-whole reasoning): the formula was manipulated to an equivalent formula, for instance, $y=4x^{-2}$ to $y=4/x^2$
- con (construction): no function family or prototype was mentioned, the formula was not an IGF: the graph was constructed through, for instance, attribute reasoning or calculating points.

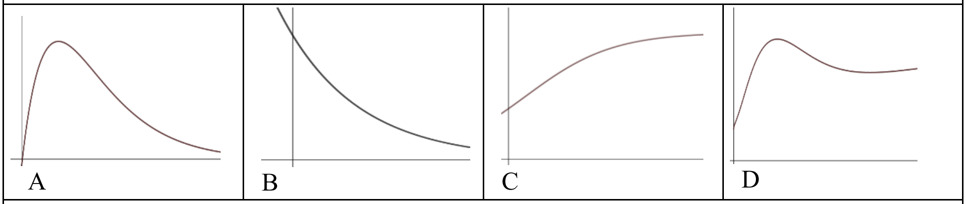
We give two examples of the encoding in Figure 3.6.

Task 3B-1: Indicate which graph(s) can fit $y=-(x+2)^4+16$			
 A	 B	 C	 D

Answer: an x^4 (pr); translated to the left, reversed and a translation (att) ; it has to be something like this (gesture a parabola); this one (A) goes to minus infinity (att) but has a positive zero and that is not possible (att) ; so, it has to be D.

The formula was an IGF because of the use of a prototype.

Task 3B-4: Indicate which graph(s) can fit $y = 500/(2 + 3 \cdot 0.75^x)$



Answer: so when x goes to infinity then it goes to 500/2=250 (att); it is divided by an ever smaller number so the result will increase (att); so it comes from beneath; at 0 it gives 100 (att); it will be positive (att), so it is this one (C).

The formula was not an IGF, because no function family or prototype was used.

Figure 3.6 Examples of the encoding of fragments of protocols of task 3

3.4 Results

3.4.1 Results of task 1

The experts' and authors' categorizations are shown in Appendix 3.1. The experts showed a great deal of agreement in their choices of categories, names of these categories, and prototypes of the categories. Only expert S used a different approach in his categorization of polynomial functions. He based his categorization on the number of turning points. The other experts all used the degree of polynomial functions. The fourth degree polynomial functions were divided into graphs with a W-form, a M-form, and a V-form (or as the experts mentioned, "increasing or decreasing"). No large differences were found on exponential functions and logarithmic functions, although some experts (P and R) made no distinction between "normal" and "reversed" graphs (for instance, $y = e^x$ versus $y = e^{-x}$ and $y = \ln(x)$ versus $y = -\ln(x)$). All experts agreed on linear broken functions and square-root functions. More differences were found in the categories of power functions, where only expert Q made distinctions based on domain and/or on concavity.

In the construction of the common expert categorization, the distances between individual categorizations and the common expert categorization were calculated. The final common expert categorization is shown in Table 3.1.

Table 3.1 Common expert categorization.

Categories:
Linear: $x + 5(4 - x)$, $\ln(e^{2x})$, $(1 - x)(2 + x) + x^2$
Parabola parabola with max: $x(9 - x)$, $-(x - 3)^2$, $(x - 5)(3 - x)$, $2x - 3(x + 2)(x - 2)$, $-(x - 1)^2 + 2(x - 1) + 6$; parabola with min: $x^2 - 7(x - 5)$, $(6 - x)^2$, $x^2 + (-x + 1)^2$
3rd degree oscillation: $(x^2 - 7)(x - 5)$, $2(x - 3)^2(x + 3)$, $2x^3 + 4x^2 - 16x$, $x^3 - 9x$, $e^{3\ln(x)}$
4th degree W-shape: $x^4 - 16x^2 - 28$, $(x^2 - 7)^2$ (6th degree W-shape: $3(x^4 - 6)(x^2 - 8)$) M-shape: $-3(x^2 - 4)(x^2 - 6)$, $2x^3(6 - x)$; V-shape: $(x + 3)^4 - 9)$, $x^2(9 + x^2)$
Exponential increasing: 4^{-3+x}, $2(\sqrt{2})^x$; decreasing: $18 \cdot 0.3^x$, 2^{6-x}, $8e^{-x}$, 10^{-2x+5}, $8/3^x$; reversed exponential: $6 - 2^x$, $100 - e^x$
Logarithmic increasing: $\ln(e^2 \cdot x)$, $1 + \log_2(x), \ln(x) + \ln(2)$; decreasing: $-\ln(x)$, $\ln(1/x)$; distractor: $1/\ln(x)$
Hyperbola(-like) hyperbola: $x(x - 1)/((x + 1)(x - 1))$, $(4x + 2)/x$, $1 - 5/(x + 1)$ power functions with negative odd power: $8x^{-3}$; with negative even power: $2/x^4$, $3x^{-2}$ slant asymptote: $x - 4/x$; two vertical asymptotes $(x^2 - 1)^{-1}$, $2/x - 3/(x - 1)$
'Roots' increasing '$\sqrt{x}$-like': $3\sqrt{x + 6}$, $2\sqrt{x} - 6$, $(100x)^{\frac{1}{2}}$; decreasing '$\sqrt{x}$-like': $4\sqrt{10 - x}$, $(2 - x)^{\frac{1}{2}} + 2$ half a circle: $\sqrt{8 - x^2}$; V-shape: $\sqrt{8 + x^2}$ power functions: exponent '$\frac{1}{3}$-like'< 1: $2\sqrt[3]{x}$, $2\sqrt[3]{x^4}/(2x)$; exponent '$\frac{1}{3}$-like'> 1: $(2x^{\frac{1}{3}})^5$; exponent '$1\frac{1}{2}$-like': $2x\sqrt{x}$

The following distances from the final categorization were found: 11, 3, 19, 20, and 15 (for P, Q, R, S, and T, respectively). The experts needed an average of 18 minutes: 23, 20, 11, 14, and 21 minutes (for P, Q, R, S, and T, respectively). For this task the experts used a lot of part-whole reasoning in order to categorize, for instance, the following formulas correctly: $\ln(e^x)$, $(1 - x)(x + 2) + x^2$, $\ln(1/x)$, $x(x - 1)/((x + 1)(x - 1))$, $(2x^{\frac{1}{3}})^5$.

From the interviews and observations we know that the experts first made a global categorization. Later they looked in greater detail and used more attributes to discriminate between the formulas. The experts described their strategy as "from simple to more complex" (expert P), "I made a preliminary categorization based on the function families with which I was brought up: with polynomial, exponential, logarithmic, power, broken, and root functions and only after this I did focus on the graphs." (expert Q), and "some I see at first sight, others only with second thoughts, like $x - 4/x$" (expert R). Most of the formulas in this task could be considered IGFs and the experts did not consider this task difficult: "not a daily task and nice to do, but not difficult" (expert T). Some experts mentioned the "things" they could instantly see from the formula, like definition domain, asymptotes, singularities, even/odd functions, infinity behavior. Some experts indicated that a more detailed categorization would be possible, but not without calculations: "In the next step I would have to make calculations; I would not trust myself to say more about this categorization off the top of my head" (expert Q).

3.4.2 Results of task 2

The results of task 2 (see Table 3.2) showed that three out of the five experts made no mistakes or only one mistake. Most mistakes were made with the formula $y = (4x + 2)/(x + 2)$. Four of our experts selected the alternative with the increasing hyperbola. From the other alternatives it could have been concluded that a distinction had to be made between an increasing and a decreasing hyperbola. Since a strict time limit of only 30 sec for one formula was used and all experts finished this task easily within this time limit, it was concluded that all the formulas that did not belong to the alternative "none of these" could be considered IGFs for the experts.

Table 3.2 Results of task 2

Participant	Number mistakes	Mistakes
P	3	$(4x+1)/(x+2)$; $7x\sqrt{x}$; $10/x^3$
Q	1	$(4x+1)/(x+2)$
R	1	$(4x+1)/(x+2)$
S	0	
T	5	$6x^2 - 2x^4$; $2x^{-4}$; $4-2x+x/4$; $(4x+1)/(x+2)$; $5x^7$

From the observations and interviews we learned that all experts first examined the 20 graph alternatives and had a global view of the formulas to get an impression of which aspects would play a role in this task and what they had to focus on. All experts read almost all graphs by mentioning a function family that fitted the graph. When performing this task, they used part-whole reasoning if necessary, recognized a function family and used attribute reasoning to discriminate between different options of the same function family. For instance, $y = 4^x - 5$, $y = 3\mathrm{e}^{-0.5x+4}$, $y = 4/2^x$ were all recognized as members of the exponential function family, attribute reasoning, like infinity behavior and reversing a prototypical graph was used to choose the correct alternative.

3.4.3 Results of task 3

In task 3 the protocols were analyzed using prototype, attribute, and part-whole reasoning. From the encoded protocols, we found that experts often started with prototypes of function families, followed by attribute reasoning.

We give four examples (pr = prototype; att = attribute reasoning; pw=part-whole reasoning):

Example 1: expert Q in task 3A-4 (third degree polynomial in Figure 3.5):

> Something with a higher degree (pr), decreasing (att), let's see; this is something that increases (att), zeroes indeed at 0, 2, and 4 (att), that looks reliable; and this at 0, 3, and 6, and that will be possible (att); and this one increases, oh, no it decreases too (att); would be a possible alternative; and this one not, it has its zeroes on the wrong side (att).

Example 2: expert Q in task 3A-6 (fourth degree polynomial in Figure 3.5):

> Let's see, fourth degree (pr), downwards (att); A. this one has no oscillations, and is only translated (att); B. is possible, where are the zeroes?, factorizing gives me $-x^2+9$ (pw), so zeroes at $x=3$ and $x=-3$ (att); C. is not possible, because when I divided by x^2 (pw) then no extra zeroes; d. when I divided by x^3 (pw), it gave me only one more zero; so it has to be B.

Example 3: expert S in task 3B-3 (exponential function in Figure 3.5):

> $100-50\cdot 0.75^x$ is an exponential; function (pr) with $y=100$ as a horizontal asymptote (att); that leaves B. and C.; it is 100 minus …., so it comes from beneath the asymptote (att), so it has to be C.

Example 4: expert T in task 3B-2 (Indicate which graph(s) can fit $y=-x(x-2)(x-4)$):

> This is a polynomial function of degree 3 (pr) and those graphs all look of degree 3 (pr); it is $-x^3$ (att), so that means these alternatives are not possible (indicated A. and B.); these two are possible but it is only this one (C.) because D. has not the correct zeroes (att).

Experts made no mistakes in this task and worked fast: see Table 3.3. However, not all formulas could be considered IGFs for the experts, as some graphs had to be constructed by reasoning about attributes. In particular, the graph of the logistic function

$y = 500/(2 + 3 \cdot 0.75^x)$ (task 3B-4) had to be constructed by all our experts and the formula $y = 6x^{-2}$ (task 3B-5) was only recognized as an IGF by expert Q ("it is an $y = 1/x^2$"). This description "it is an …" suggested that Q saw $y = 6x^{-2}$ as a member of a function family, that was indicated by a prototype $y = 1/x^2$. The other experts did not show this prototype reasoning and instead used attribute reasoning about a vertical asymptote, and positive outcomes.

Table 3.3 Time needed for task 3A and 3B, total number of mistakes, number of IGFs, and number of constructions.

Participants	Time 3A	Time 3B	Number of mistakes	Number of IGFs	Number of constructions
P	4:54 min	7:44 min	0	10	2
Q	4:16 min	2:13 min	0	11	1
R	3:56 min	6:27 min	0	9	3
S	4:02 min	2:44 min	0	6	6
T	6:16 min	2:46 min	0	9	3

In task 3A the experts could work from graph to formula. This can only be done when a function family is recognized from the graph. In example 1 and example 2 above, it is shown that expert Q recognized the graph as a prototypical graph of a polynomial function of degree 3 respectively degree 4. In Figure 3.7 it is indicated which experts started in task 3A their thinking aloud with mentioning a prototype of a function family. The other experts worked from the alternative formulas to the graph.

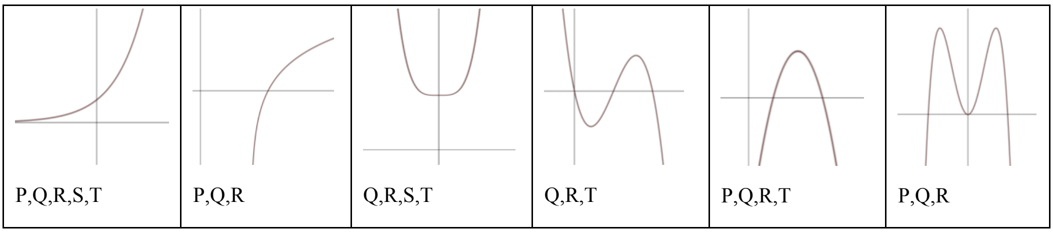


Figure 3.7 Graphs recognized as a prototype of a function family in task 3A

From the protocols we see that the experts used prototypical formulas and prototypical graphs of basic functions. They used prototypes of exponential, logarithmic, even, and polynomial of degree 2, 3 and 4 functions. Also, $y = \sqrt{a - x^2}$ (half a circle) was considered a function family. Only expert Q used $y = 1/x^2$ as a function family. Attributes that were used to discriminate between different alternatives were: increasing/decreasing of graph linked to positive/negative head coefficient, infinity behavior and horizontal asymptote, translations, vertical asymptote, number of zeroes and location of zeroes, reversing a graph, positive/negative outcomes, domain, and point of inflection.

Expert S seemed to use a lot of constructions, perhaps because a prototype or function family was not mentioned. From the protocols and results of the other tasks it was concluded that these function families were implicitly used by this expert.

From observations and interviews we learnt that the experts thought the functions used in task 3A were “easier” than those used in task 3B because they only required simple transformations. Another reason for the differences between task 3A and 3B was the amount of visual information in task 3B: “four formulas and one graph is easier to deal with than four graphs and one formula” (expert Q). Expert P mentioned that in general “it is more difficult to think from the graph than to think to the graph”. Nevertheless, all experts indicated that both tasks required the same knowledge elements: namely, linking visual features of the graphs and features of the formulas.

3.5 Conclusions and discussion

3.5.1 Conclusions

The first aim of the current research was to describe experts’ repertoires of IGFs. We hypothesized that experts would use categories to organize their knowledge of graphs and formulas. The experts’ results in task 1 showed that the categories they constructed were very similar and also that the category descriptions were similar. These descriptions were closely related to the function families of basic functions that are taught in secondary school: linear functions, polynomial functions, exponential and logarithmic functions, broken functions, and power functions. Only expert S used descriptions containing numbers of turning points for the polynomial functions. Therefore, a common categorization could be constructed. The distances between the individual categorizations and the final categorization varied from 3 to 20. Many of these differences could be explained by the absence of subcategories. For instance, some of the experts did not distinguish between increasing and decreasing exponential graphs or between parabola with a maximum or with a minimum. However, the experts’ performances in task 2 confirmed that they could recognize these differences between subcategories as they made almost no mistakes in this task.

The time the experts needed to perform this categorization task varied from 11 to 23 minutes. When taking about 20 minutes to categorize 60 cards, the experts needed only 20 seconds per card to read, to recognize, to compare with others, and to group formulas with similar graphs. This meant that there was almost no time for the construction of new, unknown graphs. Some of the formulas, like $y = 1/\ln(x)$, $y = (x^2 - 1)^{-1}$, $y = x - 4/x$, $y = 2/x - 3/(x - 1)$ were categorized in a category with a single formula, often with a mention of some attributes, but without a graph. Therefore, it was concluded that the experts used the function families of the basic functions from secondary school to organize their categories of IGFs : linear functions; 2nd , 3rd and 4th degree polynomial, exponential, logarithmic and root functions with, in every function family, a distinction between increasing and decreasing; broken linear function; power functions x^n , with n odd/even, and $n = p/q$ with $p > q, p < q$. This should come as no surprise, since we used predominantly formulas of basic functions from the secondary school curricula. The experts were brought up with these categories, as they indicated in the interviews. They showed through their high proficiency that they had truly internalized this categorization of basic functions. The formulas seemed to be complex enough to capture the proficiency of the experts, as some formulas could not be instantly visualized or were not correctly categorized.

The second aim of the current study was to describe what experts attend to when linking formulas to graphs of IGFs. The recognition process when working from formulas to graphs can be well described using the Barsalou model of Figure 3.2. It is shown in Table 3.3 that in recognizing IGFs, the experts often started with prototypes. This prototype reasoning was, when necessary, followed by attribute reasoning. For instance, $y = -2x(x - 2)(x - 6)$ is recognized as a prototypical "x^3", which is "reversed" and has zeroes at 0, 3, and 6; $y = \log_2(x + 3)$as a log translated to the left; $y = -(x + 2)^4 + 6$ as an "x^4", reversed and translated; $y = \sqrt{6 - x^2}$ as "half-a-circle". These examples were in line with the findings of Schwarz and Hershkowitz (1999), who found that proficient students used prototypes as levers for handling other examples and showed greater understanding of (critical) attributes.

The experts also recognized prototypical graphs for well-known function families, as they showed in task 2 and task 3A. For well-known function families there seemed to be little difference between working from formula to graph and working from graph to formula. When working with IGFs, the experts' concept images that were triggered by the given formula or given graph, seemed to contain equivalent formula(s), graph(s), attributes of graphs and of formulas, function family with prototypes, formulas of other functions in this function family.

In order to elicit experts' attribute reasoning, all attributes the experts used in task 3 were gathered: translation to the right/left and above/below, stretching horizontal or vertical, reversion (often indicated by reasoning about negative head coefficient), infinity behavior (with horizontal asymptotes), increasing/decreasing, number and location of turning points, location and number of zeroes, positive/negative, domain, point of inflection, and vertical asymptotes.

Particular attributes seemed to be linked to particular function families. As shown in task 3, these connections could work both ways: from function families to salient attributes of graphs and from graphs with salient attributes to function families. These salient attributes of a function family are characteristic of the members and prototypes of the function family. For instance, a vertical asymptote was directly linked to logarithmic functions or broken functions. And, when confronted with power functions with $n = p / q$, some instantly started with a focus on domain and concavity. For the different function families in our research, the experts used salient attributes: limited domain was linked to root functions, power functions and logarithmic functions; vertical asymptotes were linked to logarithmic functions and broken functions; horizontal asymptotes were linked to exponential functions and broken functions; symmetry was linked to even polynomial functions.

Experts used attributes appropriate to the tasks. For instance, when they had to link formulas to global graphs (tasks 1 and 2), they paid no attention to the factor 7 in $y = 7x\sqrt{x}$, or to the term 3 and term 1 in $y = 4^{-3+x}$. But when parameters influenced the global shape of the graph, these parameters were given ample attention. For instance, the minus signs of the head coefficient in $y = -x^4 + 9x^2$ and in $y = 2\sqrt{8-x}$ which reversed the prototypical graphs were directly noticed and mentioned. When more detailed graphs were requested, as in task 3, the experts again only used those attributes that were needed for the task. For instance, they did not mention anything about the factor 0.1 in $y = 0.1x^2$ or about the term 12 in the formula $y = x^6 + 12$, because these were positive numbers. But when the task demanded it, the experts quickly noticed the attributes and values needed to graph the formulas. For instance, the experts instantly recognized the different locations of the zeroes in $y = x(3-x)(x-6)$ and $y = -2x(x+3)(x+6)$. These findings show that the experts worked efficiently and did not pay attention to "what is normal" (Chi et al., 1981; Chi, 2011).

The experts sometimes had to show their abilities in algebraic manipulation. As expected, they had no problems with this aspect of part-whole reasoning: this was shown in, for instance, $y = 6x^{-2}$ (in task 3B), and $y = 8x^{-3}$, $y = x(x-1)/((x+1)(x-1))$, and $y = x^2 + (-x+1)^2$ (in task 1).

These results show that experts' processes of recognition of IGFs can be described using the model in Figure 3.2: with prototypes, supported by attribute and part-whole reasoning.

A Barsalou model for recognition of IGFs

The current findings highlight the two highest levels of recognition of the framework for strategies in graphing formulas (Kop et al., 2015). We defined formulas at these levels as IGFs, instantly graphable formulas. We described the experts' repertoires of IGFs and described what experts attended to in recognizing IGFs. We showed that the experts used prototypes and attribute reasoning in recognizing IGFs and found how particular attribute and value sets were linked to particular function families. For instance, given a logarithmic formula such as $y = 1 + \log_3(2x+4) - 3$, a prototype $y = \log_3(x)$ or $y = \log(x)$ was instantly identified and attribute reasoning (translation, domain $x > -2$, and/or vertical asymptote at $x = -2$) resulted in a graph. We also found that for function families of basic functions, the experts could easily work from graph to a formula. Given a graph, they instantly recognized a function family that fitted the graph. For instance, a graph with attributes like domain $x > a$, a vertical asymptote at $x = a$ and concave down was instantly identified as a logarithmic function. This implies that the Barsalou model based on Schwarz and Hershkowitz in Figure 3.2 can be expanded with linkages between attribute and value sets, prototypes and function families and with linkages from graph to attributes, prototypes, and function families. In Figure 3.8, for some of the function families in the experts' categorizations (logarithmic, polynomial with degree 2, exponential, and broken functions), a prototype is described using attributes and values; for other exemplars of the function family salient attributes are indicated.

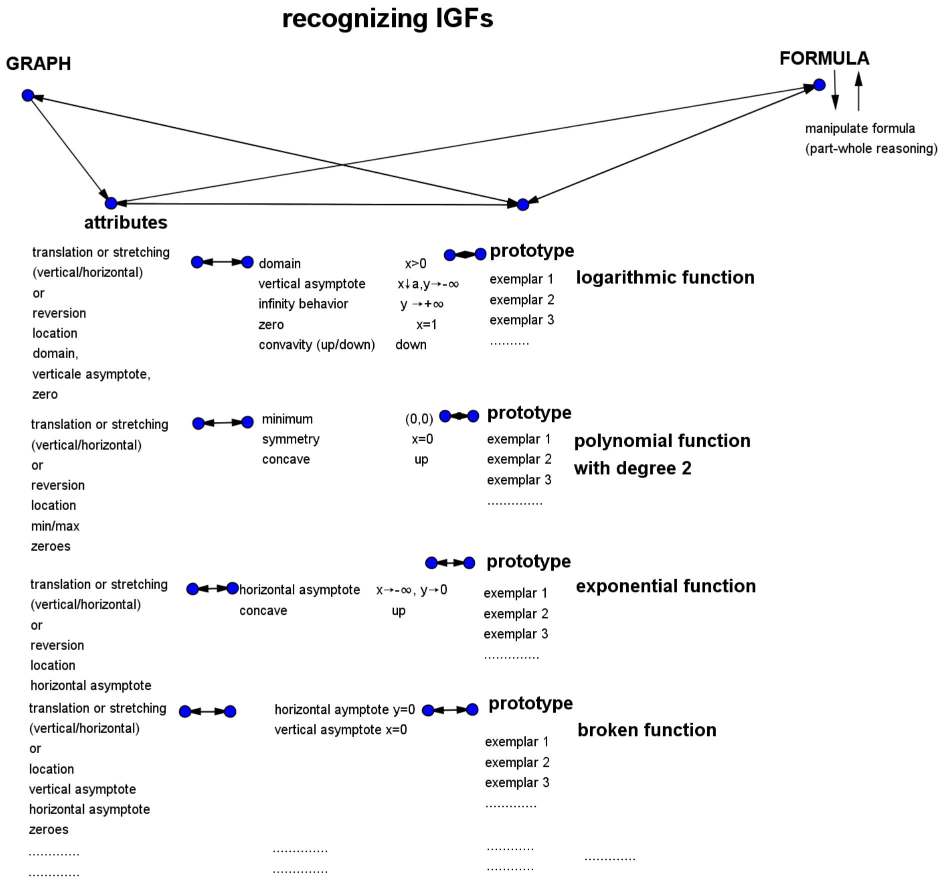


Figure 3.8 A Barsalou model based on Schwarz and Hershkowitz with function families and their salient attributes

3.5.3 **Global properties in graphing formulas**

The experts in our study focused on attributes and values that influenced the global shape of the graph. For instance, a parameter that reversed the prototypical graph of $y = x^4$ was given ample attention, like -2 in $y = -2x^4$, but a parameter that resulted in only a small change of the prototypical graph was not mentioned, such as 0.1 in $y = 0.1x^4$. In their attribute reasoning, the experts focused on attributes and values that gave a great deal of information about the whole graph. These attributes can be considered the global growth properties of Slavit's classification of function properties (Slavit, 1997). Starting with these global properties is considered to be more efficient in graphing formulas than using local properties (Even, 1998; Slavit, 1997).

In the current study we found that the experts used a set of attributes and values that differ from Slavit's global properties, partly because Slavit's focus was more on the function concept, whereas our focus was on the relation between formula and graph. Several global properties Slavit used, such as integrability and invertibility, were not mentioned at all by our experts. Based on the current results, we suggest that relevant global properties for recognizing IGFs may be symmetry, infinity behavior (including horizontal and slant asymptotes), vertical asymptotes, domain, increasing/decreasing on intervals, sign (reverse), and concavity. For local properties, we suggest zeroes, turning points, points of inflection, and individual points.

3.5.4 Suggestions for further research and teaching

In discussing their ideas about graphs and formulas, the experts used an ample repertoire of descriptions: "a valley", "it goes in the right direction", "it has to go downwards", "it runs flat", "tails go to minus infinity", "this one has no oscillations", "a reversed …..", "it goes to infinity", "it goes up", "it comes from below", "in infinity it is …", "this one is only positive", "log to the right", "an oscillation downwards", "a $-x^3$".

These descriptions show that the experts often did not use the formal math attribute/property concepts but used both pictures of the whole graph and action language such as "it (the graph) runs ……". People talk ubiquitously about abstract concepts using concrete metaphors (Barsalou, 2008). Metonymies and metaphors are necessary for efficient communication and in the learning of mathematical concepts (Presmeg, 1998; Zandieh & Knapp, 2006). Further research is necessary to find out how these experts' metonymies and metaphors can be helpful in the efficient teaching of graphing formulas.

A repertoire of IGFs is necessary for graphing formulas. Eisenberg and Dreyfus (1994) wrote about the need for a repertoire of basic functions and knowledge of the characteristics of the representations of these functions. Slavit (1997) speaks about "property noticing", the ability to recognize and analyze functions by identifying the presence or absence of these properties and the need for a "library" of functional properties. Our findings show how experts used prototype and attribute reasoning for graphing formulas and so give an impression of an expert "library" of properties. Figure 3.8, a Barsalou model based on Schwarz and Hershkowitz, shows how for IFGs these function families, prototypes, attributes, and part-whole reasoning are integrated in the experts' concept images. Our findings may be helpful to further describe the recognition and identification of objects, forms, key features, and dominant terms used in Pierce and Stacey's algebraic insight (Pierce & Stacey, 2004; Kenney, 2008). Not only in graphing formulas but also when using CAS or graphical calculators one needs this algebraic insight. For instance, Heid et al. (2013) showed how solving the equation $\ln(x) = 5\sin(x)$ required knowledge of the characteristics of function families of both formulas $y = \ln(x)$ and $y = 5\sin(x)$ and the ability to link the graph images to the formulas (Zbiek & Heid, 2011).

The results of this study can be relevant for teaching algebra and in particular functions. Students continue to experience difficulties with seeing the relationship between algebraic and graphical representations, although graphing technology can support students' understanding in linking representations of functions (Kieran, 2006; Ruthven, Deaney, & Hennessy, 2009). In order to further improve education, we first need a domain-specific knowledge base (De Corte, 2010). Expertise research can provide such a knowledge base (De Corte, 2010; Campitelli & Gobet, 2010; Stylianou & Silver, 2004). The current findings show what knowledge experts used in recognizing IGFs: they used the basic functions to organize the function families, used prototypes to handle other exemplars of function families, and used prototypes and attributes to link graphs and formulas of function families. In secondary school curricula much attention is paid to basic functions, in particular to linear and quadratic functions. Our study suggests that only learning and practicing basic functions is not enough to become proficient in linking the formulas and graphs of functions. Students need to know how to handle parameters in formulas and need opportunities to integrate their knowledge of prototypes and attributes of function families into well-connected hierarchical mental networks. Besides such a knowledge-base for recognition, students need heuristic methods, like splitting formulas and qualitative reasoning, when recognition falls short (Kop et al., 2015).

For graphing formulas one has to be able to "read" algebraic formulas. Further research is necessary to investigate whether graphing formulas indeed improve symbol sense, in particular algebraic insight and how graphing formulas can be effectively and efficiently taught to students.

Appendix 3.1:

Five experts' categorizations and the researcher's categorization with category names and prototypes

P. 23:22 min	Q. 20:28 min	R. 11:25 min	S. 14:25 min	T. 21:00 min	***Researcher's categorization***
Linear: $ax+b$	Straight lines: $ax+b$	Linear: $ax+b$	Linear $y=x$	Linear functions	Linear Increasing/ decreasing
Degree 2: ax^2+bx+c	Parabola with max: $-x^2$ Parabola with min: x^2	Degree 2: ax^2+bx+c	1 turning point $y=x^2$	Degree 2	Parabola with max and with min
Polynomials: $\sum_{k=0}^{n} a_k x^k$ (defined on domain)	Degree 3 (odd): x^3	Degree 3: ax^3+bx^2+cx+d	2 turning points: $y=x^3-3x$	Degree 3	Degree 3 increasing
	Degree 4, decreasing: $-x^2(x^2-1)$ Degree 4, increasing: $x^2(x^2-1)$	Degree 4: $ax^4+bx^3+..$	3 turning points: $y=(x^2-1)^2$ 5 turning points $x^2(x^2-1)\cdot(x^2-2)$	Degree 4, with W-shape Degree 4 without W-shape	Degree 4 with W-shape M-shape V-shape Degree 6 with W-shape
$(ax+b)^k/(cx+d)^n$	Hyperbola: $1/x$ Quotient functions with more than 1 vertical asymptote: $1/(x^2-1)$	Broken functions	Vertical and horizontal asymptotes $y=1/x$ Vertical and slant asymptotes $y=x-4/x$ 2 vertical asymptotes: $y=1/(x^2-1)$	Linear broken: $(ax+b)/(cx+d)$	Hyperbola and Power function with higher negative odd exponent 2 vertical asymptotes Slant asymptote
Power function with negative exponent	Even hyperbola-like: $1/x^2$	Negative exponent: ax^{-n}		Power function	Power function with higher negative and even exponent
Can be transformed to $b\cdot g^x+c$ HA ∞ HA at $-\infty$	Exponential increasing: e^x Exponential decreasing: e^{-x}	Exponential function: $a\cdot b^x+c$	Exponential: $y=2^x$	Elementary exponential function	Exponential increasing HA $x\to-\infty$ And decreasing (HA $x\to\infty$) Reversed exponential: $6-2^x$, $100-e^x$
Logarithmic function with transformation	Logarithmic increasing: $\ln(x)$	Logarithmic: $a\log(x)+c$ (included $1/\ln(x)$	Log: $y=\log_2(x)$	Elementary logarithmic function	Logarithmic increasing and decreasing

(increasing/decreasing and $1/\ln(x)$)	Logarithmic decreasing: $-\ln(x)$				
Power function, function with broken exponent, root function with transformation: $c\sqrt{ax+b}+d$	Roots, domain to the left: $\sqrt{-x}$ Root, domain to the right: $\sqrt{x}$	Root function: $\sqrt{ax^2+bx+c}$	$y=\sqrt{x}$	Elementary root function, transformed $y=\sqrt{x}$	Roots increasing '$\sqrt{x}$-look-like' and decreasing '$\sqrt{x}$-look- like'
$\sqrt{ax^2+bx+c}$	Power function even $\sqrt[4]{x^2}$		Half a circle	Root	Half a circle: $y=\sqrt{8-x^2}$ V-shape: $y=\sqrt{8+x^2}$
Broken power function, not transformed from basic function	Odd power function: $\sqrt[3]{x}$ Broken exponent, defined to the right: $x\sqrt{x}$	Broken exponent: $ax^{\frac{p}{q}}$		Power function, positive exponent, no asymptote	Power function with broken exponent, concave up/down
	Various: $1/\ln(x)$)		Apart: $1/\ln(x)$)	Rest $1/\ln(x)$; $(x^2-1)^{-1}$; $3(x^4-6)\cdot(x^2-8)$	distractor: $1/\ln(x)$

CHAPTER 4

Promoting insight into algebraic formulas through graphing by hand

This chapter is based on: Kop, P. M., Janssen, F. J., Drijvers, P. H., & van Driel, J. H. (2020a). Promoting insight into algebraic formulas through graphing by hand. *Mathematical Thinking and Learning*, 1-20. https://doi.org/10.1080/10986065.2020.1765078

Abstract

Student insight into algebraic formulas, including the ability to identify the structure of a formula and its components and to reason with and about formulas, is an issue in mathematics education. In this study, we investigated how grade 11 students' insight into algebraic formulas can be promoted through graphing formulas by hand. In an intervention of five 90-minute lessons, 21 grade 11 students were taught to graph formulas by hand. The intervention's design was based on experts' strategies in graphing formulas, that is, using a combination of recognition and qualitative reasoning, and on principles of teaching complex skills. To assess the effect of this intervention, pre-, post-, and retention tests were administered, as well as a post-intervention questionnaire. Six students were asked to think aloud during the pre- and post-tests. The results show that all students improved their abilities to graph formulas by hand. The think-aloud data suggest that the students improved both on recognition and reasoning and give a detailed picture of how students used recognition and qualitative reasoning in combination. We conclude that graphing formulas by hand, based on the interplay of recognition and qualitative reasoning, might be a means to promote students' insight into algebraic formulas.

4.1 Introduction

Research has shown that students in grades 11 and 12, and even beyond secondary school, have persistent difficulties with algebra in general, and with dealing algebraic formulas and making sense of them in particular (Arcavi, 1994; Arcavi et al., 2017; Ayalon et al., 2015; Chazan & Yerushalmy, 2003; Drijvers et al., 2011; Kieran, 2006; Hoch & Dreyfus, 2005, 2010; Oehrtman et al., 2008). The students lack symbol sense, which is defined as the very general notion of “when and how” to use symbols (Arcavi, 1994). Symbol sense has several aspects, such as the ability to read through algebraic expressions, to see the expression as a whole rather than as a concatenation of letters, and to make rough estimates of the pattern that would emerge in a graphical representation (Arcavi, 1994; Pierce & Stacey, 2004). Drijvers et al. (2011) describe symbol sense as complementary to basic skills. Symbol sense involves strategic work with a global view and an emphasis on algebraic reasoning, whereas basic skills involve procedural work with a local focus and an emphasis on algebraic calculations. Pierce and Stacey (2004) use algebraic insight to capture the symbol sense involved in using computer algebra software. This algebraic insight concerns identifying structure through the recognition of objects, key features, dominant terms and simple factors, knowing the meaning of symbols, and the ability to link representations (Pierce & Stacey, 2004).

In this study we aimed at this one aspect of symbol sense, namely, insight into algebraic formulas, that is, the ability to “look through a formula.” More specifically, we viewed insight into algebraic formulas as including the abilities to recognize the structure of a formula and its components and to reason with and about a formula. Structure in algebra has been defined by Hoch and Dreyfus (2010) as a broad analysis of the way an entity is made up by its parts. Structure sense includes abilities such as seeing an algebraic expression as an entity, recognizing the expression as a previously met structure, dividing the entity into sub-structures, and recognizing the connection between structures. In this study we focused on functions of one variable and their Cartesian graphs. We chose to use graphing formulas by hand, without technology, as a means to promote students’ insight into formulas. In this article, this graphing formulas by hand will be called *graphing formulas*.

Many studies about symbol sense and graphing are about the role of technology like graphic calculators to promote students’ symbol sense (Arcavi et al., 2017; Drijvers, 2003; Hennessy et al., 2001; Heid et al., 2013; Kieran & Drijvers, 2006; Philipp et al., 1993; Yerushalmy & Gafni, 1992). In some of these studies, the need for by hand activities has been stressed (Arcavi et al., 2017; Kieran & Drijvers, 2006), but to our knowledge there are no recent studies that investigate effects of graphing by hand on students' symbol sense and this study might fill this gap. We investigated how graphing formulas might be learned by students and designed an intervention consisting of a series of lessons on graphing formulas, in grade 11 (16- and 17-year-old pre-university students) to enhance students' insight into algebraic formulas. In this way, the current study contributes to the understanding of how recognition, reasoning and its interplay involved in graphing formulas may foster students' insight into formulas.

4.2 Theoretical framework

Graphing formulas is a complex task for students. In this section, we elaborate on the theoretical principles underlying our educational design. First, the literature about symbol sense and graphing is discussed. Next, we describe the nature and content of the knowledge base students need for graphing formulas. Finally, we discuss how this knowledge base might be addressed in student tasks, using the literature on teaching complex skills.

4.2.1 Symbol sense and graphing

To promote insight into formulas, we had two arguments for focusing on graphing formulas. First, we targeted insight into formulas that are often used in grade 11 textbooks, like $y = 4\sqrt{10 - x}$, $y = 2(x - 3)^2(x + 3)$, $y = (x + 3)^4 - 9$, $y = (4x + 2)/(x + 3)^2$, $y = x\mathrm{e}^{-x}$, $y = \ln(x - 3)$, so, we needed a general domain, in which many different formulas could be addressed. Second, in literature, it has been recommended to use realistic contexts and multiple representations to give meaning to algebraic formulas (Kieran, 2006; Radford, 2004), and to learn about functions (Arcavi et al., 2017; Kieran, 2006; Janvier, 1987; Leinhardt et al., 1990; Moschkovich et al., 1993). However, besides linear and exponential functions, it is in general difficult to link formulas to realistic context, except in mathematical modeling. Therefore, we chose for using representations, in particular for linking formulas to their graphs.

Graphing tools such as graphic calculators are helpful for learning about functions and their multiple representations (Hennessy et al., 2001; Kieran & Drijvers, 2006; Heid et al., 2013; Philipp et al., 1993; Yerushalmy & Gafni, 1992). However, Goldenberg (1988) found that students established the connection between formula and graph more effectively when they did graphing by hand than when they only performed computer graphing. Therefore, we chose the context of graphing formulas by hand to promote students' insight.

In linking formulas to graphs, covariational reasoning comes into play. Covariational reasoning concerns coordinating two varying quantities while attending to how they change in relation to each other (Thompson, 2013; Carlson et al., 2002). While the focus often is on quantities in real-life situations, algebraic functions with "imagining running through all input-output pairs simultaneously and so reason about how a function is acting on an entire interval of input values" are also included (Carlson et al., 2002). Covariational reasoning often focuses on the global graph and five levels of development have been described: from the idea that change in one variable depends on change in another variable, to paying attention to the direction of change, to paying attention to the amount of change, to considering average rate with uniform increments of the input variable, to the instantaneous rate of change for entire domain (Carlson et al., 2002; Oehrtman et al., 2008). It has been argued that such covariational reasoning is critical in supporting student learning of functions in secondary and undergraduate mathematics (Carlson et al., 2002; Confrey & Smith, 1995; Oehrtman et al., 2008; Thompson & Carlson, 2017). Students have difficulties with this reasoning. This was shown by Carlson, Madison, and West (2015), who found that students were not able to select the correct graph (out of five alternatives) of $f(x) = 1/(x-2)^2$, indicating, according to the authors, that students were not able to reason "as the value of x gets larger the value of y decreases, and as the value of x approaches 2, the value of y increases." Such reasoning about functions requires a global perspective on a function, that is, seeing the function as an entity or object (Confrey & Smith, 1995; Even, 1998; Gray & Tall, 1994; Oehrtman et al., 2008). This may be hindered by another commonly used perspective, namely, seeing a function as an input-output machine (a given x-value is linked to a certain y-value). The latter view is considered a pointwise, process, or correspondence perspective. A global perspective is more powerful and gives a better understanding of the relation between formula and graph, but a pointwise approach is needed to construct initial meaning (Even, 1998). To learn graphing formulas, students have to learn to take a global perspective on functions and to use the first three levels of covariational reasoning, in particular paying attention to the direction of change and the global amount of change of a function (concavity).

4.2.2 Expertise in graphing formulas: recognition and reasoning

To investigate what is needed to master a complex skill, it has been recommended to examine expert behavior (Kirschner & Van Merriënboer, 2008; Schoenfeld, 1978). In expertise research, it has been established that for effective and efficient problem solving one needs recognition, and reasoning when recognition falls short (Berliner & Ebeling, 1989; Chi et al., 1981). In our previous studies, we described experts' recognition and strategies involved in graphing formulas (Kop et al., 2015; Kop et al., 2017). Five experts from different backgrounds, but all holding a master's or PhD in mathematics, were selected to investigate expertise in graphing formulas: three mathematicians who worked at Dutch universities, one mathematical textbook writer who was also a mathematics teacher in upper secondary school, and one who worked at the Dutch Institute for Testing and Assessment. Because all had more than 10 years of experience in work which often required them to graph formulas, we considered them experts in graphing formulas (Kop et al., 2015; 2017).

To describe experts' thinking processes for graphing formulas, different levels of recognition were formulated: the formula can be instantly visualized as a graph or is recognized as a member of a function family of which the global graph is known; the formula can be decomposed into sub-formulas of function families; some characteristics of the graph are instantly recognized but not the whole graph; there is no recognition at all (Kop et al., 2015). These levels of recognition can be linked to Mason's (2003) levels of attention, in which he described how attention can shift from seeing essential structure to gazing at the whole and not knowing how to proceed. For recognition, a repertoire of basic function families that can be instantly visualized by a graph (Eisenberg & Dreyfus, 1994) and knowledge of features to describe graphs are needed (Slavit, 1997). Kop et al. (2017) found that experts' repertoires of basic function families resembled the basic function families taught in secondary school, like exponential, logarithmic, and polynomial functions. Experts seem to have linked prototypes of these function families to a set of critical graph features. For instance, a prototypical logarithmic graph has a vertical asymptote, only positive *x*-values as a domain, and is concave down. Experts use their repertoire of basic function families as building blocks in working with formulas to decompose complex functions into simpler basic ones and to read characteristic graph features from formulas (Kop et al., 2015, 2017).

When experts graph more complex formulas and instant recognition falls short, they start reasoning about, for instance, infinity behavior, in/decreasing of a function, and weaker/stronger components of a function, but they hardly use calculation of points and/or derivatives. In short, our previous studies suggest an interplay of recognition and reasoning being the backbone of the expertise at stake. We give five examples to illustrate experts' recognition and reasoning. (1) Sketching $y = 2\sqrt{x+6}$: "It is a root-function translated to the left." (2) Sketching $y = -2x(x-3)(x-6)$: "It is a polynomial function of degree 3, reversed because of $-x^3$, and zeroes at 0, 3, 6." (3) Sketching $y = 100 - 50 \cdot 0.75^x$: "It has *y*=100 as an horizontal asymptote, 100 minus …, so, it comes from beneath to the asymptote; when *x* is very negative, it is 100 minus very large outcomes, so y-values will be very negative." (4) Sketching $y = x - 4/x$: "I can sketch $y = x$ and $y = 4/x$, now I have to subtract the graphs, here (with large values of x) it is almost $y = x$, when x is a little bit larger than 0 y is very negative, etc. (sketch the graph). (5) Sketching $y = 500/(2 + 0.75^x)$: "When x goes to infinity then it is 500/2=250; 500 dividing by a decreasing number, so outcomes increase; it is always positive, and when x goes to minus infinity it is almost 0." (Kop et al., 2015; 2017)

The interplay between recognition and reasoning is visible when experts use prototypical graphs of function families. For example, "a root-function translated" in (1); "a polynomial of degree 3, reversed" in (2); "decomposing a formula, graphing both sub-formulas, and compose these sub-graphs" in (4) (Kop et al., 2017; Schwarz & Hershkowitz, 1999). These examples show that experts can start with prototypical graphs and use reasoning about transformations, about characteristics, about composing sub-formulas to finish the graph. Sometimes, experts only recognize a key graph feature and have to use more reasoning to complete the graph. For example, in (3), the horizontal asymptote was instantly recognized. It also possible that there is no recognition, then experts start strategic exploration of the graph. For example, in (5) the expert started reasoning about infinity behavior of the function. Experts' reasoning is often qualitative of character, that is, global reasoning, using global descriptions without strict proofs, and ignoring what is not relevant. We illustrated this experts' qualitative reasoning in the five examples above. In their reasoning experts ignored the factor 2 when sketching $y = 2\sqrt{x + 6}$ (1) and $y = -2x(x - 3)(x - 6)$ (2), the factor 50 in $y = 100 - 50 \cdot 0.75^x$ (3). Ignoring what is not relevant is an aspect of adaptive reasoning and an indication of expertise (Chi et al., 1981; Chi, 2011). Global reasoning is found when exploring parts of a graph, for instance, infinity behavior of the function in (3) "x is very negative, it is 100 minus very large outcomes, so y-values will be very negative" and in (5) "when x goes to minus infinity it is almost 0." Global descriptions were used in (2) "reversed" and in (3) "it comes from beneath to the asymptote".

In literature the importance of qualitative reasoning with its focus on the global shape of the graph and ignoring what is not relevant has been addressed. Leinhardt et al. (1990) spoke about qualitative interpretation of graphs to gain meaning about the relationship between the two variables, and their pattern of covariation. In physics and physics education, qualitative reasoning is used to describe essential entities and processes and to provide the necessary grounding for a deep and robust understanding of quantitative models (Bredeweg & Forbes, 2003; Forbes, 1996). Friedlander and Arcavi (2012) used the term qualitative thinking in their framework for cognitive processes involved in algebraic skills. Qualitative thinking is about predicting and interpreting results without calculation and/or manipulation skills and strict proofs. Experts use this qualitative reasoning also in their communication with students. For example, Thompson (2013) described how an experienced teacher added two sub-graphs using blank axes to keep students away from calculations, focusing on an estimation of the sum-graph, and using qualitative reasoning in the discussion with the class, with descriptions like “it is less negative,” “how negative,” “it will get lower.” However, this qualitative reasoning, with its ignoring what is not relevant and its focus on the global shape of the graph, is often used implicitly and hardly taught explicitly in school (Duval, 2006; Leinhardt et al., 1990).

Experts’ recognition and reasoning in graphing formulas inform us about “what to teach”: students should learn a repertoire of basic function families, with prototypes and key features, for recognition and students should learn experts’ reasoning, with its qualitative character, using global descriptions, ignoring what is not relevant, and without strict proofs. In the next section we address literature on complex skills to formulate design principles (DP) about how to teach graphing formulas, based on recognition and reasoning.

4.2.3 Teaching complex skills

Although graphing formulas is a well-described task, it can also be considered a complex task, because functions may vary from basic functions to very complex ones. In this section, we outline a social constructivist approach to teaching graphing formulas as a complex skill. In this approach, students learn component knowledge and skills in the context of more complex whole tasks, with adaptive support and students are invited to articulate and reflect on their own problem-solving processes (De Corte, 2010).

Complex cognitive skills consist of many constituent skills, which have to be integrated and coordinated. In education, a part-task approach is often used: all constituent skills are taught separately and in succession, and only at the end are students confronted with the complexity of the whole task. This results in students having difficulties in integrating and coordinating all the constituent skills (Kirschner & Van Merriënboer, 2008).

Instead of the part-task strategy, a whole-task-first approach is recommended: students learn skills and knowledge in the context of entire tasks (Collins, 2006; Kirschner & Van Merriënboer, 2008; Merrill, 2013; Van Merriënboer, Clark, & De Croock, 2002). Of course, students cannot immediately perform an entire task without help. Therefore, it is recommended to support student learning processes in different ways (Kirschner & Van Merriënboer, 2008; Merrill, 2013; Van Merriënboer et al., 2002). In the context of graphing formulas, the whole-task-first approach means that students are confronted from the start with the full complexity of graphing formulas; that is, they have to deal with different kinds of functions and strategies (DP 1). In order to support students, help is provided in different ways: through modeling (that is, showing expert thinking processes to students), examples, overviews, sub-questions, and reflection questions (DP 2) (Kirschner & Van Merriënboer, 2008; Merrill, 2013; Van Merriënboer et al., 2002).

Landa (1983) described the importance of general thinking methods or meta-heuristics that are needed to use one's skills and knowledge in problem situations. Pierce and Stacey (2007) indicated the importance of teaching students the habit of starting with the question "What do I notice about this expression which may be important?" We call this "questioning the formula," which can be considered a meta-heuristic (Arievitch & Haenen, 2005; Landa, 1983). In graphing formulas, students should learn to internalize and automatize the habit of questioning the formula (DP 3).

In the current study, we used these three design principles to design an intervention on graphing formulas, with the aim to promote insight into formulas of functions of one variable. The following main research question guides the study:

How can grade 11 students' insight into algebraic formulas be promoted through graphing formulas?

4.3 Method

In this section, we subsequently describe the intervention, including the tasks that were used in the teaching, the participants, the instruments used in the pre-test, post-test, and retention test, and the data analysis.

4.3.1 Intervention

The intervention took five lessons of 90 minutes. Each day, the lesson started with a short plenary discussion (max 10 minutes) with general feedback on the students' work, reflection on the tasks, and modeling of expert thinking processes. After the plenary, the students worked in pairs or groups of three, studied their personal feedback and the written elaborations on the tasks given by the teacher, and discussed strategies and solutions for the whole tasks. The teacher visited each group at least once during a lesson to give further explanations and coaching. At the end of a lesson, all pairs and groups handed in their work for personal feedback which focused on the reflection questions, for which students had to construct their own examples.

The intervention started with a whole class discussion about the levels of recognition; this was to introduce the meta-heuristic “questioning the formula” (DP 3). The aim was that students would develop the habit of asking themselves questions like: “Do I instantly know the graph?”, “Do I recognize a function family?”, “Can I decompose the formula?”, “Do I recognize graph features?”, “Can I do some strategic search for, for instance, infinity behavior?” At the end of the intervention, but before the post-test, 18 of the 21 students voluntarily attended a longer plenary discussion of 30 minutes in which they discussed strategies for graphing several formulas.

The tasks used in the teaching were formulated as whole tasks, reflecting the levels of recognition and the meta-heuristic “questioning the formula”: task 1 and 2 concerned recognition of basic functions and aimed to develop a knowledge base of function families with their characteristic features and to deal with simple transformations; task 3 concerned the decomposition of formulas and the composition of sub-graphs through qualitative reasoning; task 4 concerned the instant recognition of key graph features; and task 5 was about strategic exploration of parts of a graph, through qualitative reasoning. We now give some examples of the tasks.

> Task 1 required students to match formulas of basic function $y = \sqrt{x},\ y = x^3$, $y = 0,5^x$, $y = \ln(x),\ y = |x|$ to their graphs. Task 2 was based on Swan (2005): Describe the differences and similarities between the graphs of the pairs of functions like $y = 2\sqrt{x} - 4$, $y = 2\sqrt{x-4}$ and $y = -3^x,\ y = 3^{-x}$. In task 3, the function $y = \sqrt{x}\,(3x-6)$ had to be graphed by multiplying the graphs of the sub-functions $y = \sqrt{x}$ and $y = 3x - 6$. Task 4 was inspired by Burkhardt and Swan (2013) and Swan (2005), and concerned the recognition of graph features: What features of the given graph can be instantly read from the given two equivalent formulas $y = (x-4)^2 - 1$ and $y = (x-5)(x-3)$?
> Task 5 concerned reasoning about parts of a graph (part-graph exploration). For instance, what happens to the y-values of the functions $y = 0.6^x \cdot x^{60}$, $y = 52.7/(1 + 62.9 \cdot 0.692^x)$, when $x \to +\infty$? Choose $y \to +\infty$; $y \to a \neq 0$; $y \to 0$; $y \to -\infty$

For each task, help was provided, and a reflection question was added. For instance, in task 2 (about recognizing transformations of basic functions) students could choose to use GeoGebra, and/or to study worked-out examples for help. After each whole task, a reflection question was posed, in which students were asked to construct three new examples to demonstrate the principles of the whole task. Constructing examples is a means to stimulate students to reflect (Watson & Mason, 2002).

4.3.2 Participants

The intervention was held in the first author's grade 11 mathematics B class, a regular class of 21 pre-university students, who were 16 or 17 years old. Mathematics B is a course that prepares students in the Netherlands for university studies in mathematics, science and engineering. In regular education in the Netherlands, students learn about linear, quadratic and exponential functions in grade 8 and 9. In grade 10, the graphic calculator is introduced and power, rational, logarithmic functions and the derivative are the most important topics. In grade 11, further exploration of derivatives and rules for differentiation are taught, together with solving calculus problems (e.g., optimization, tangent, and parameter problems) using algebraic manipulation and the graphic calculator. In this school, students were used to working together on tasks in an open space, as there was only one small room for plenary instruction available, which could be used once a week.

4.3.3 Data collection

We collected all individual student responses to three written tests: the pre-test, the post-test, and the retention test, all of which had two similar tasks: a graphing task and a multiple-choice task that focused on recognition (indication of the total time: 25 min). The formulas used in the three tests, though not the same, were comparable in structure and difficulty. To avoid a learning effect from the tests, the students' work was not returned to them. The period of four months between the post-test and the retention test, including a holiday period, was considered long enough to prevent learning effects.

The three written tests demonstrated the students' competencies to graph formulas but gave only limited information about their recognition and reasoning. Therefore, more detailed information about the students' thinking processes was needed: six students were asked to think aloud during the pre-test and post-test, when working on the graphing task. These interviews were videotaped and transcribed. Thinking aloud is not expected to disturb the thinking process and should give reliable information about problem-solving activities (Ericsson, 2006). As it was possible that the effect of the intervention would depend on students' previous mathematics performance, the six students were selected on the basis of their earlier mathematics performances during the school year: two high-achieving (S and K), two more than average-achieving (A and M), and two average-achieving students (Y and I).

In a post-intervention questionnaire, the students were asked to report their ideas about the series of lessons. Six questions were posed: whether they had improved their skills in graphing formulas (1), whether they had learned to use more strategies (2), whether their recognition of graph features had improved (3), whether their recognition of formulas that could be instantly graphed had improved (4), whether they could switch their strategy more often (5), and whether they used the meta-heuristic "questioning the formula" more often when graphing formulas (6). In each question, the students were asked to indicate, on a scale of 1 to 4, to what extent they agreed with the statement. In two open questions, the students were invited to make remarks about the series of lessons and their learning during these lessons. The first author (teacher) kept a logbook with lesson plans, and short descriptions of the plenary discussions and other aspects of the student's learning.

4.3.4 Graphing task and multiple-choice task in the tests

The first task used to investigate the students' insight into formulas was "Draw a rough sketch of the following functions ... ." We selected seven simple and seven more complex functions, all of which could appear in the students' mathematics textbooks. The simple functions aimed to assess the students' repertoire of basic function families and their reasoning using prototypical graphs, transformations and/or function family characteristics. Examples of these simple functions are $y = \sqrt{6-2x}$ and $y = (x^2-4)(x^2-6)$. The more complex functions, like $y = \sqrt{x}(x-2)(x-6)$ and $y = 3x\sqrt{x+2}$, aimed to assess the students' recognition of graph features and their part-graph exploration.

To assess the students' recognition abilities, a multiple-choice task with 21 alternatives (20 graphs and one "none of these"; see Figure 4.1) was also used. For 16 functions, the students were asked to match the formula to the global shape of the graph. The following are examples of functions that were used: $y = 2x(x-9)$, $y = x^2(6-x^2)$, $y = 4^x - 5$, $y = 2\sqrt{8-x}$, $y = -2\sqrt{x}$. Figure 4.1 shows four of the 20 graphs that were used as alternatives.

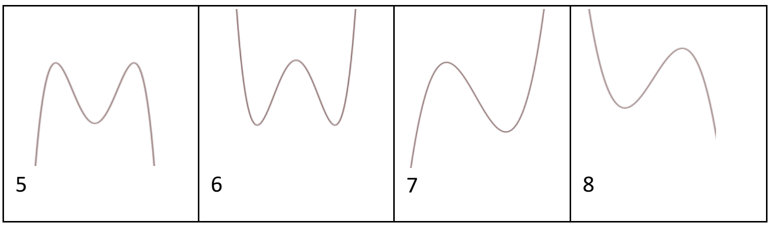


Figure 4.1 Some alternatives in the multiple-choice task

4.3.5 Data analysis

For the analysis of the graphing task, the graphs in all tests were graded as correct, partly correct, or not correct, resulting in a score of 1, 0.5, or 0. We graded a graph as partly correct when many but not all aspects of (the construction of) the graph were correct. For example, when the graph of $y = -2x(x-2)(x-5)$ had zeroes at $x = 2$ and $x = 5$, and the direction of the "oscillation" was correct, but the graph failed to show the zero at $x = 0$, or when the sub-graphs of $y = x^2\mathrm{e}^x$ ($y = x^2$ and $y = \mathrm{e}^x$) were correctly graphed but mistakes were made in the composition of the sub-graphs. For each student, the total score, the score on simple functions, and the score on complex functions were calculated. For the multiple-choice task in all tests, each item was graded as correct (score 1) or incorrect (score 0), resulting in a total score on the multiple-choice task.

To compare the scores on the graphing task and multiple-choice task of the pre-test, post-test, and retention test, the mean scores and standard deviations were calculated. A paired *t*-test with the effect size (Cohen's *d*) for each task was calculated to determine differences between pre-test and post-test results (short-term effect) and differences between pre-test and retention test (long-term effect).

The thinking-aloud protocols of the graphing task of the six students were transcribed and time was recorded. The transcripts were cut into units of analysis which contained crucial steps of students' recognition and reasoning (Schwarz & Hershkowitz, 1999). To analyze students' insight, we used categories with descriptions of the experts' strategies in graphing formulas (see examples in Theory section): combinations of recognizing and reasoning involving function families, involving key graph features, and part-graph exploration. The encoding of the thinking-aloud protocols was done according the instructions in Table 4.1.

Table 4.1 Codebook for thinking-aloud protocols

Encoding	description	example
P1 (a prototypical graph)	If a function family has been recognized (mentioned) and a prototypical graph and/or (qualitative) reasoning (with transformations and/or characteristics) are used to sketch the graph.	Sketching $y = 2\sqrt{x+8}$: "it is a root-function, translated to the left." (factor 2 can be ignored). Sketching $y = \sqrt{8-x}$: "it is a reversed root-function, and edge point is (8,0)"
P2 (two prototypical graphs)	If two function families have been recognized (mentioned), two sub-formulas are graphed and the two sub-graphs are combined using qualitative reasoning.	Sketching $y = x\mathrm{e}^{-x}$: "decompose it into $y = x$ and $y = \mathrm{e}^{-x}$, and multiplying the two sub-graphs, when x is very large e^{-x} is almost 0 and stronger than x, so $y \approx 0$; when x is very negative y will be very negative."
F (key graph feature)	If the graph has not been recognized but a key graph feature has been recognized.	"It has a vertical asymptote at x=3" or "it has zeroes at ..."; but not when calculating the y-intercept.
PG (part-graph)	If the graph has not been recognized and parts of the graph are explored using qualitative reasoning.	"When x is large, y is … (infinity behavior)," or "in the neighborhood of x=3 …"
C (calculation)	If more than two points of the graph are calculated, or if a derivative is calculated, or brackets in a formula are expanded.	

The units of analysis were encoded by the first author and checked by another researcher, which resulted in recoding of 10% of the transcripts. When the student succeeded in making a correct (rough) sketch of the formula (score of 1), using P1, P2, F, PG, we interpreted this as a sign of insight into this formula, resulting in an insight-score of 1. However, if the student used the calculation of more than two points and/or the derivative (C), we said that the student had no insight in this formula, resulting in an insight-score of 0. If the graph was partly correct and sketched via P1, P2, F, PG, we considered this as showing "some insight," resulting in an insight-score of 0.5.

Below, we illustrate these encodings with two examples (other examples in the Result section).

Student A sketching $y = -(x-3)^4 - 5$ correctly with insight-score 1, used a prototypical graph:

> Looks like parabola; turning point is in $(3, -5)$ (P1); parabola with a maximum (P1)

Student A sketching $y = (x^2+6)/(x^2-4)$ correctly with insight-score 1, decomposed the formula into two parabola, graphed both sub-graphs, then used graph features (vertical asymptotes and symmetry) and part-graph exploration about infinity behavior of the function and the graph's behavior in the neighborhood of the vertical asymptotes. Only two points of the graph were calculated, so the insight-score was 1.

> (tries to manipulate the function $(x-2)(x+2)$); no, this does not work; first decomposing; (graphed both parabolas) (P2); when x is very large than y is close to 1 (PG); when x=2 no outcomes, so a vertical asymptote (F); and also at $x=-2$; when x is a little bit larger than 2 than the denominator is very small and the dominator relative large (PG); the larger x will be the smaller the outcomes will be (PG); when $x=-2$ this will be the same (F); when x=1 y is 7/−3 is about −2; when $x=-1$ I get the same ; when x=0 it is −1.5; when x is just under 2, the denominator is negative and the dominator is very large, so is goes to minus infinity (PG); at $x=-2$ the same, because of symmetry (F).

To analyze the post-intervention questionnaire, the mean scores were calculated for each question and an inventory of the remarks about the series of lessons was made.

4.4 Results

The results of the graphing and multiple-choice tasks of the pre-test, post-test, and retention test are described first, then we report the results of the six thinking-aloud students on the graphing task, and finally the results of the post-intervention questionnaire and fragments of the teacher's logbook.

4.4.1 Graphing tasks

The results of the graphing tasks gave information about the students' abilities to graph formulas. For a first impression of the effect of the intervention, we compared the mean scores in the pre-test, post-test and retention test. We distinguished between the simple and the complex functions. Table 4.2 shows that the mean total score in the pre-test was 2.95 out of 14, with a standard deviation of 2.42. The post-test scores were higher, with a mean total score of 9.21. In the retention test the mean score dropped to 6.97. A similar pattern was found for basic functions and complex functions.

The paired t-tests that were used to calculate the differences between the scores in the pre-test and post-test and between the pre-test and retention test showed that all score differences were significant with $p<0.01$. Cohen's d, used to quantify these differences were rather large. See Table 4.2.

Table 4.2 Results of graphing task in pre-, post-, and retention-test compared

	pre-test Mean (SD)	post-test Mean (SD)	t-value & p-value pre-post test	Cohen's d	retention-test Mean (SD)	t-value & p-value pre-retention test	Cohen's d
Total	2.95 (2.42)	9.21 (2.58)	$t(20) = 13.00$; $p < .001$	2.50	6.97 (3.35)	$t(14) = 3.48$; $p = .001$	1.38
Simple	2.42 (1.42)	6.19 (1.26)	$t(20) = 10.73$; $p < .001$	2.87	4.87 (1.80)	$t(14) = 5.40$; $p < .001$	1.56
Complex	0.74 (1.41)	3.02 (1.71)	$t(20) = 9.51$; $p < .001$	1.45	2.10 (1.85)	$t(14) = 2.46$; $p = .028$	0.83

4.4.2 Multiple-choice tasks

The results of the multiple-choice tasks gave information about the students' recognition of basic function families and graph features. The results on these tasks showed the same pattern as on the graphing tasks: the scores on the 14 items were low in the pre-test, increased substantially in the post-test, and decreased slightly in the retention test. Table 4.3 shows that the differences were significant and that the effect sizes were rather large.

Table 4.3 Results of multiple-choice task in pre-, post-, and retention-test compared

	pre-test Mean (SD)	post-test Mean (SD)	t-value & p-value pre-post test	Cohen's d	retention test Mean (SD)	t-value & p-value pre-retention test	Cohen's d
Total	2.95 (2.29)	10.01 (2.79)	$t(20) = 10.17$; $p < .001$	2.22	8.07 (3.33)	$t(14) = 4.65$; $p < .001$	1.20

4.4.3 Thinking-aloud protocols on the graphing task

First, we give an overview of the results of the six students who thought aloud during the graphing task, then, we portray the recognition and reasoning of two representative students (student M and the high-achieving student K) in the pre-test and post-test. These examples illustrate their insight into formulas, that is, their recognition and reasoning when graphing formulas. We end this section with some remarks about the results of the four other students.

Table 4.4 shows the scores of the six thinking-aloud students on the graphing task in the pre-test and post-test (on simple and complex functions) and the time they needed to finish these tasks. In addition, their scores on the retention test are indicated. For instance, in the pre-test, student K had a of 4 out of 7 on simple functions and a score of 4 out of 7 on complex functions; their insight-score was 3 on both simple formulas and complex formulas; K used in 6 graphs prototypical graphs and in 5 graphs part-graph exploration.

Table 4.4 Results from thinking-aloud protocols: Scores, insight-scores, kinds of reasoning, time

	Pre-test				Post-test				Retention test
Student	Score (s+c)[1]	Insight score (s+c)	Kinds of reasoning	Time (min)	Score (s+c)	Insight-score (s+c)	Kinds of reasoning	Time (min)	Score (s+c)
S	3.5+4	3+4	[2]P12:6 F:3 C:0 PG:4	>16:42[3]	7+6	7+6	P12:8 F:2 C:0 PG:5	17:48	7+7
K	4+4	3+3	P12:6 F:1 C:5 PG:5	38:25	7+6	7+6	P12:12 F:2 C:0 PG:2	16:00	5+2
M	2+0	2+0	P12:2 F:6 C:0 PG:1	34:20	7+3	7+3.5	P12:10 F:4 C:0 PG:1	16:00	5.5+2
A	5+0	4+0	P12:4 F:1 C:3 PG:5	16:25	7+2	7+2.5	P12:12 F:1 C:0 PG:1	17:12	6+2
I	2+0.5	1+0.5	P12:4 F:6 C:0 PG:9	27:50	7+3	5+3	P12:5 F:6 C:2 PG:9	21:00	5+1.5
Y	2+1	2+0	P12:5 F:0 C:3 PG:7	>30:55[4]	5.5+3.5	5.5+3.5	P12:10 F:2 C:0 PG:3	18:22	4+1

[1] score on simple functions (s) and on complex function (c).
[2] P12=using prototypical graphs and/or composition of 2 sub-graphs; F=recognizing key graph feature; PG=part-graph exploration.
[3] For 11 graphs in pre-test, [4] for 12 graphs in pre-test.

Table 4.4 confirmed the higher scores in the post-test and retention test in comparison with the pre-test and the differences in scores between simple and complex formulas, as found in Table 4.2. Table 4.4 shows that the time needed in the post-test was much shorter than in the pre-test, that the scores and insight-scores in the post-test were higher than in the pre-test, and that the scores and insight-scores were closely related. The latter indicates that calculations were hardly used in successfully graphing formulas. Table 4.4 also shows that the high-achieving students did relatively well on complex formulas in the pre-test and only missed one graph in the post-test. The results show that most students used more prototypes of function families in the post-test. In the retention test, only student S graphed all formulas correctly, but the scores of the other five students were still higher than in the pre-test.

To illustrate the student's insight, we portray the recognition and reasoning of two representative thinking-aloud students: student K as a high-achieving student, and student M as one of the other four students. In the pre-test, student M had great difficulties graphing formulas: M only recognized the graphs of root-functions and features like zeroes of polynomial-functions and vertical asymptotes but had a limited repertoire of reasoning. This resulted in a score of only 2 correct graphs out of 14 (only $y = 3\sqrt[4]{x} + 2$ and $y = \sqrt{6 - 2x}$). Some citations illustrate their thinking processes and insight.

M sketching $y = (x - 3)^2 - 9$ (insight-score 0); after calculating a point and part-graph reasoning with "when *x* is increasing then *y* … ," M could not sketch the graph:

> …At $x = 3$ $y = -9$. (After some time) the larger *x* is, the larger *y*, so it increases (PG). It is a parabola. (M stopped talking for a while; after a couple of minutes) I do not know how to proceed. Encoding: PG.

M also had problems with sketching $y = \ln(x - 3)$ (insight-score 0); after recognizing a translation, M did not know the shape of the ln(*x*)-graph, and tried to construct the graph via the inverse function (but did not succeed):

> …graph of $\ln(x)$, that is translated 3 to the right (F) (M did not use this, instead writes $\log_e(x - 3) = \log(e)/\log(x - 3); e^y = x - 3$). This is an asymptote (F); *x* cannot be 3; ….; when $y = 0, x - 3 = 1$ (drew point (4,0) and stopped). Encoding: F.

In the post-test, M's insight had improved, resulting in a score of 10 out of 14. Some citations to illustrate these improvements:

M sketching correctly $y = 30 \cdot 0.92^x + 40$ (insight-score 1), used a prototypical decreasing exponential graph and a translation, and described globally the function's infinity behavior:

> Decreasing exponential function (sketched a prototypical graph) (P1); 40 above (P1); when *x*=0, *y*=70; later approximately 40 (PG). Encoding: P1,PG.

When sketching $y = -2x(x - 2)(x - 5)$ correctly (insight-score 1), M recognized the zeroes, and used qualitative reasoning when exploring the function's behavior at *x*=1 and when ignoring the factor 2 in $-2x$:

> …goes downwards (F); zeroes at 0, 2, 5 (F). At *x*=1, it is negative (PG). Encoding: F,PG.

M showed "some insight" into $y = (x^2 + 6)/(x^2 - 4)$(insight-score 0.5), as M did not indicate the horizontal asymptote; the function's behavior in the neighborhood of *x*=2 was explored:

> …asymptotes at 2 and -2 (F); zero at $\sqrt{6}$; no, no zeroes, because x^2 cannot be negative; when *x* is smaller than 2, then it is positive here, and negative here, so it is negative (PG); when *x* is a bit larger than 2, positive here, positive here, so positive (PG); the same for -2 (F). Encoding: F,PG.

Although the high-achieving student K scored 8 correct graphs out of 14 in the pre-test, K then had problems with recognizing basic function graphs. However, K was able to compensate this lack of recognition through her reasoning abilities and the calculation of many points. We give two examples to illustrate this: when K did not know the ln(x) graph and when K could not read the zeroes from $y = (x-2)(x-6)$.

K, sketching $y = \ln(x-3)$ (insight-score 1), did not recognize the shape of a logarithmic function, but used qualitative reasoning about the inverse function to sketch the graph correctly:

> I do not know the ln-graph anymore. When $x-3=0$, then …… When $x-3=1$, then $y=0$, so $x=4$. At x-as the x-axis is intersected. When x is increasing then y increases, so the graph increases (PG). When x is negative … (thinking). Because something to the power of e ($e^{\cdots}$) does not give negative y-values (PG). So, $x-3$ cannot be negative; the graph only exists from x=3, larger than 3 (PG). So, at x=3 a tangent (means asymptote) and outcomes smaller when x is in the neighborhood of 3 (PG). Encoding: PG.

K sketching $y = \sqrt{x}(x-2)(x-6)$ correctly but with insight-score 0; K decomposed the formula, but was then unable to sketch the graph of the parabola $y = (x-2)(x-6)$ using recognition and reasoning, as K did not recognize the zeroes and needed the calculation of more than two points of this parabola; however, K showed their reasoning abilities when constructing a correct graph by multiplying the two sub-graphs using qualitative reasoning:

> First expanding the brackets: $y = \sqrt{x}(x^2 - 8x + 12)$,….sub-function is parabola with minimum and root function, $\sqrt{x}$ goes like this (P1); when x is negative, this part remains empty (left y-axis) (P1); at x=0, parabola gives =+12; (sketched an incorrect parabola through (0,0)); …; (calculation of points, (1,5) and (4, −8) (C) (noticed that parabola is incorrect and calculated more points of parabola; (2,0), (4, −4), (6,0)); so, parabola goes like this (correct parabola); between 2 and 6 (parabola) negative, so, positive (root) times negative gives negative (P2), and more negative than −4; …; it goes through (1,5);…so, I expect that the graph progressively increases because of $\sqrt{x}$ (P2) and that is looks like a parabola; at x=0, y=0, that means that between 0 and 1 something strange happens; it goes like …..$\sqrt{x}$; ….; (sketched a correct graph). Encoding: P1,P2,C

In the post-test, K's recognition of basic functions had improved and K still used their reasoning abilities, resulting in a score of 13 out of 14. Some citations to portray their insight into formulas:

When sketching $y = 2\sqrt{5-x}$ correctly (insight-score 1), K ignored the factor 2:

> "times -1; exists for $x \leq 5$ (P1); so, starts at x=5 (P1), and from there is goes like this". Encoding: P1

When sketching $y = 2x\sqrt{x+6}$ correctly (insight-score 1), K decomposed the formula and used part-graph reasoning in the composition of the two sub-graphs:

> 2x goes like this (P1); $\sqrt{x+6}$ goes like this (sketch) (P1); here it is 0; here negative, here 0, and after this it is steeper (P2). Encoding: P1,P2.

K sketching $y = 30/(2 + 6 \cdot 0.9^x)$ correctly with insight-score 1, gesturing the sub-graph of the denominator, ignoring the factors 2 and 6 and reasoning about infinity behavior of the function:

> 0.9^x goes like this (P1); $6 \cdot 0.9^x$ and $2 + 6 \cdot 0.9^x$ (P1) like this (gestured correct graph); …30/….; 30/2 goes to 15; that means a horizontal asymptote (F); 30 divided by an ever increasing number (looks at the negative x-axis) becomes smaller, goes to 0 (PG). Encoding: P1,F,PG.

These examples of student K illustrate how the two high-achieving students (K and S) were already able to reason with and about formulas in the pre-test, but had problems with the recognition of the prototypical graphs and characteristics of basic function families. In the post-test, their recognition had improved, and they were able to combine their recognition and reasoning more effectively and efficiently, resulting in more insight into formulas. Also, the four other students had problems with recognizing basic function families and their characteristics in the pre-test, but their reasoning then was very limited, as was illustrated by citations of student M and Table 4.4. In the post-test, these four students showed insight in almost all the simple formulas. However, they still had problems with complex formulas. Although they showed more insight as they were able to make the first steps (e.g., decomposing into sub-formulas and graphing correct sub-graphs), they had difficulties composing the two sub-graphs and/or finding and combining all relevant graph information. Two examples to illustrate these problems:

Student Y graphed $y = -x^4 + 2x^2$ partly correct (insight-score 0.5); Y missed that the graph of x^4 is "flatter" than the one of x^2 in the neighbourhood of $x = 0$:

> Adding both (P2); I split the function; a parabola "to the power of 4" will run like this (P1) (sketches the graph of $y = -x^4$); $2x^2$ goes like this (P1); this is not nice, you have to add

them; the ‘to the power of 4’ is stronger than ‘ to the power of 2’, so, ….; it goes through 0; then adding; this one ($-x^4$) is stronger, thus it goes under this one” (sketched a parabola-shaped graph with maximum)(P2). Encoding: P1,P2

Student I graphed $y = x + e^{-x}$ partly correctly (insight-score 0.5); the sub-graphs were correct, but the composition was incorrect:

> e^x goes like this; -*x*, so ($y = e^x$) is reversed over *y*-axis (P1): it becomes smaller and is not negative; the larger *x*, the smaller *y*; e^{-x} is stronger; $y = x$ goes like this (P1) (sketched two correct sub-graphs); when $x=-1$ it is positive; when *x* is more positive, than e^{-x} becomes smaller and *x* larger, but e^{-x} is stronger, so, the outcomes are smaller and negative (P2)(sketched a graph beneath the x-axis for large values of *x*). Encoding: P1,P2

4.4.4 Post-intervention questionnaire and teacher’s logbook

In the post-intervention questionnaire, the students indicated, on a scale of 1 to 4, whether they thought they had improved their skills in graphing formulas (mean score 3.1), that they used more strategies (mean score 3.2), that they had improved their recognition of graph features (mean score 3.3), and that they used “questioning the formula” more often (mean score 3.0). However, the scores on “more formulas could be instantly graphed” and “being able to switch strategy” were lower: 2.8 and 2.4 respectively. In their answers on the open questions about the series of lessons and their learning during these lessons, the students indicated they thought their recognition of basic functions and graphs had improved, that they could visualize formulas (of basic functions) faster, and that they “understood” formulas better.

Also, the teacher’s logbook confirmed the progress in the students’ insight during the intervention. To illustrate this, we provide some quotations from the teacher’s logbook. During the first lesson: “The pre-test was not motivating for the students, but after some time they are working on the teaching tasks.” During the second lesson: “The task about transformations is hard for these students and costs more time than needed.” During the fourth lesson: “In the groups, I heard their reasoning with ‘this one goes like this (with gesturing)’ and ‘when *x* is very large, then …’.” During the last lesson: “The high-achieving students show more interest in the plenary discussions than usual. They seem to be challenged by these tasks. One of the students indicated that they thought these lessons (in the intervention) are different from regular lessons: ‘we now use global reasoning (referring to qualitative reasoning); it is fun this kind of reasoning’. In a discussion the students showed their abilities to reason qualitatively when discussing the graph of $y = 10\sqrt{6 - x} + 3$. One of the students had drawn a global graph on the whiteboard, using (6,3) as starting point, and sketched a reversed root-graph (i.e., a root graph to the left). Another student wondered what had to be done with the 10 in the formula. The first student responded 'hardly anything, only when one wants to compare the graph with 10 and the graph with, for instance, 8. However, the graph with -10 is reversed, so very different. The same is true for the 3 in the formula: 3, 4, 5 does not matter, but -3 does matter.' A third student then explained this fact by referring to the scaling of the vertical axis."

4.5 Discussion and conclusion

The current research aimed at promoting insight into algebraic formulas, an important aspect of symbol sense. To foster grade 11 students' insight, we chose to teach experts' strategies in graphing formulas, which could be described through a combination of recognition and reasoning (Kop et al., 2015, 2017). In this study, we designed an intervention of five lessons of 90 minutes, focusing on the recognition of basic function families and of graph features, and on qualitative reasoning, and investigated whether students' insight was enhanced. The pre-test results of the written tests showed that the students had problems with graphing formulas and the thinking-aloud protocols suggested a lack of recognition and reasoning skills, which resulted in time consuming calculations and many incorrect graphs. The lack of recognition was confirmed by the results of the multiple-choice test.

In the post-test, the results of the written tests showed large improvements. The thinking-aloud protocols of six students showed how their recognition and reasoning skills had improved. All six students showed insight into formulas, as they could now recognize function families and use these in their reasoning. However, unlike the two high-achieving students S and K, the other four students still had problems in graphing the complex functions. Although these four students showed more insight into complex functions, using decompositions into sub-functions and graphing these sub-functions correctly, they often made mistakes in the composition of the two sub-graphs and/or in finding and combining all relevant information. The results of the post-intervention questionnaire suggest that the students themselves thought that their skills in graphing formulas had improved, that they used more strategies and more recognition, and that they had more insight into formulas, as they indicated that they understood formulas better.

In the retention test, the scores on the graphing task and multiple-choice task were, as expected, lower than in the post-test. Still, the scores were higher than in the pre-test. This suggests a long-lasting effect of the intervention, in particular on simple functions.

The findings of the current study suggest that through this intervention in which students were taught to graph formulas using recognition and qualitative reasoning, students improved their insight into formulas, that is, the ability to identify structure of a formula and to reason with and about formulas.

Before we address the study's limitations and reflect on the intervention, we discuss the findings. In the current study we chose to use graphing formulas to foster students' insight into formulas, in contrast to other approaches that focus on manipulations and/or structures of expressions. Graphing formulas is a small domain in algebra, which makes it more possible for students to learn experts' strategies. However, graphing formulas is also a rich domain, as it can involve all kinds of functions and involves aspects which are important in learning about functions: the relation between two major representations of functions, formulas and graphs, allowing students to give meaning to abstract algebraic formulas (Kieran, 2006), and the need of both a global and a local perspective on functions to learn about the process and object duality of functions. The results of the thinking-aloud protocols reveal that all students started to use experts' strategies, although only high-achieving students were able to correctly graph complex formulas. Students used insight into formulas to graph formulas, but hardly used algebraic manipulations even if these would be more convenient, for example, when graphing $y = -x^4 + 2x^2$. The results of the questionnaire and the logbook suggested that the graphing tasks in the intervention were challenging and encouraged students to engage in algebraic reasoning. We believe that our strategy to select a small domain in algebra and to focus on just reading through formulas and making sense of formulas might explain a part of the positive students' results in this study.

Our approach differs from regular approaches as well as from innovative approaches to learn about algebraic formulas as it was based on a systematical analysis of experts' strategies in which the two elements, recognition of function families and key graph features and qualitative reasoning, both play an important role. Regular approaches often focus on manipulation of algebraic expressions (Arcavi et al., 2017; Schwartz & Yerushalmy, 1992), and use graphing tools, for example, the graphic calculator, to explore function families and to work on calculus problems. In comparison to regular approaches, our intervention paid more attention to the recognition of function families and graph features, to part-graph exploration, and to the reasoning with and about formulas. In innovative approaches, graphing tools have been used to learn to reason about functions using the structure of the formula, for instance, the composition and translation of graphs (Schwartz & Yerushalmy, 1992; Yerushalmy & Gafni, 1992; Yerushalmy, 1997), about the role of parameters (Drijvers, 2003; Heid et al., 2013), and about special function families (Heid et al., 2013). Pierce and Stacey (2007) suggested highlighting the formula's structure and key features when considering graphs in classroom discussions. Friedlander and Arcavi (2012) developed a framework comprising different cognitive processes and activities, including qualitative thinking and global comprehension, and formulated small tasks in which components of their framework had been worked out. In comparison to these innovative approaches, our intervention paid more attention to the systematical teaching of thinking tools: a repertoire of basic function families, the recognition of function families and key graph features, and qualitative reasoning. In the designed intervention these aspects were taught in an integrated way via a task centered approach with adaptive support.

In the current study several levels of recognition and several aspects of qualitative reasoning were distinguished. Often recognition is treated as a dichotomous variable: there is recognition or there is no recognition. In our approach we use different levels of recognition: complete recognition and instantly knowing the graph, recognizing a member of a function family, decomposing the formula into manageable sub-formulas, perceiving key graph features, no recognition. These levels of recognition can be linked to Mason's (2003) levels of attention, in which has been described how attention can shift from seeing essential structure to gazing at the whole and not knowing how to proceed. An essential aspect in our approach was the explicit focus on qualitative reasoning. The importance of this kind of reasoning and its omission in mathematics curricula has been stressed by Leinhardt et al. (1992), Goldenberg, Lewis and O'Keefe (1992), Yerushalmy (1997), and Duval (2006), who have indicated that qualitative reasoning could support the construction of meaning and understanding through its global focus. To our knowledge, this idea has never been applied in concrete and systematic teaching approaches. In our approach students learned to use global descriptions and to ignore what is not relevant, when composing two sub-graphs (after decomposing a formula into two sub-formulas) and when exploring parts of a graph, for instance, infinity behavior. We recommend paying more attention to explicit teaching of qualitative reasoning in grade 11. We expect that in other domains of algebra, such as solving equations, qualitative reasoning might help students to become more proficient in algebra, as it might enable students to ignore what is not relevant and to focus on the structure of formulas/equations.

In the designed intervention not only attention has been paid to recognition and to qualitative reasoning, but also explicit attention is paid to the *interplay* between recognition and qualitative reasoning. In problem solving, recognition determines the problem space within which via certain heuristics can be searched for a solution (Berliner & Ebeling, 1989; Chi et al., 1981). In the intervention, each whole task was related to one of the levels of recognition (see intervention in Method section 4.3), and attention was paid to the reasoning needed to sketch the graph, starting from this level of recognition. This approach enables students to use different ways to graph a function like $y = 30/(2 + 6 \cdot 0.9^x)$: in the post-test we found students who decomposed this function into two sub-functions ($y = 30$ and the exponential function $y = 2 + 6 \cdot 0.9^x$), but also students who used part-graph exploration (infinity behavior and/or the function is increasing), and/or the calculation of the *y*-intercept. These examples illustrate how a correct graph can be produced via different levels of recognition in combination with different reasonings and that insight into formulas can be described as an interplay between recognition and reasoning. The analysis of the thinking-aloud protocols showed how students' insight into formulas could be described with the recognition of a function family and (qualitative) reasoning about transformations and/or family characteristics, the decomposition of a formula into two sub-functions and the composition of two sub-graphs through qualitative reasoning, the recognition of key graph features, and the qualitative reasoning about parts of a graph. Although in other domains of mathematics, like in modeling and solving equations, insight into formulas might consist of different aspects, our descriptions might be helpful in describing insight and in designing education to promote insight into formulas in these domains.

Insight in the interplay between recognition and reasoning can contribute to a better knowledge about covariational reasoning in the context of algebraic functions. Graphing formulas by hand is closely related to this kind of covariational reasoning. Both are about how a function is acting on an entire domain, have a focus on global graphs and use qualitative reasoning. The current study showed that the use of function families with their prototypical graphs and characteristics is crucial in graphing formulas. However, Moore and Thompson (2015) have problematized what they called static shape thinking, that is, seeing a graph-as-a-wire, and associating shapes with function properties. Previous studies about expert behavior in graphing formulas have showed that experts often use their repertoire of function families (Kop et al., 2015, 2017). Eisenberg and Dreyfus (1994) and Slavit (1997) have indicated that students need such a repertoire of basic function families and function properties. The pre-test results of our study showed that before the intervention students lacked a repertoire of function families that could be instantly visualized by a graph. As a consequence, graphing formulas required too much reasoning of these students. Post-test results showed that students had improved their recognition of basic function families with their prototypical graphs and characteristics, which could be used as building blocks in their reasoning. The results of our study suggest that students' covariational reasoning might improve if they can use such repertoire of function families to reason with prototypes.

4.5.1 Limitations of the study

A limitation of the study is that only one class was involved, and no comparison group was included. As the results were positive, we would recommend involving more students and other teachers in a future study to provide stronger evidence that graphing formulas in this way does indeed promote students' insight into algebraic formulas. We suggest also including students and teachers from other countries in a future study, as we expect that difficulties with insight into algebraic formulas are not exclusive to students in the Netherlands. In the current study, insight into formulas was studied in the context of graphing formulas. We expect that there might be some transfer from insight into formulas from this domain of graphing formulas to other domains of algebra, such as solving algebraic problems and solving equations. More research is needed to explore whether students who have learned insight into formulas via graphing formulas will be able to use this insight when working on other algebraic problems that are related to graphs (e.g., discussing the number of solutions of a given equation).

In the pre-test, the students needed more time than expected for the graphing task. This might be the reason for the poor scores on the multiple-choice task in the pre-test, as many students did not have enough time to work on that task. From the thinking-aloud protocols, we conclude that some students had problems interpreting the graphs in the multiple-choice task, as they thought that the x-axis and y-axis were drawn instead of vertical and horizontal asymptotes. We suggest to explicitly indicate the asymptotes in the figures and illustrate this via an example in the task description. The whole task on transformations of basic functions (task 2) took much more time than planned, and the students often needed the help provided by the teaching material. The whole tasks on the composition of two sub-graphs and on part-graph exploration by qualitative reasoning (task 3 and 5) needed, as planned, extra modeling by the teacher, as this kind of reasoning was new to the students. The meta-heuristic of "questioning the formula" was at the core of this series of lessons and was demonstrated more than once. In the post-intervention questionnaire, the students indicated that they had started to question formulas. However, this was often very implicit, as the thinking-aloud protocols showed.

Some aspects of the series of lessons deserve more attention in the future. On each level of recognition, only one whole task with a reflection question was formulated, because of time constraints (5 lessons). With more time available, we would follow Kirschner and Van Merriënboer's (2008) suggestion to use more variability in the whole tasks (more whole tasks on each level of recognition) with more practicing of the integration and coordination of all sub-skills. To improve reflection, the implementation of cumulative reflection tasks, which promote reflection on the current task and all previous tasks, might be considered. In the current study students had problems with graphing polynomial functions, like with $y = -x^4 + 2x^2$, but not when zeroes could easily be read from the formulas, like with $y = -2x(x - 3)(x - 6)$. When graphing $y = -x^4 + 2x^2$, students used qualitative reasoning to compose two sub-graphs, after decomposing the formula into sub-formulas $y = -x^4$ and $y = 2x^2$, which gave them much trouble and incorrect graphs. These findings suggest to pay more attention to polynomial function families and to incorporate small manipulations of algebraic formulas, for instance, to rewrite $y = -x^4 + 2x^2$ into $y = x^2(-x^2 + 2)$, which would enable students to find zeroes of polynomial functions.

4.5.2 Conclusion

This study portrays how students might learn insight into formulas, that is, the ability to "look through a formula", to recognize the structure of a formula and its components, and to reason with and about a formula. Graphing formulas requires students to recognize the structure of formulas and to reason with and about formulas. Therefore, our teaching focused on using function families as meaningful building blocks and on using qualitative reasoning. Students often see formulas on an atomic level, that is, paying attention to every number and variable, which means that students cannot see the wood before the trees: they do not recognize any structure (Davis, 1983).

The current study showed how students learned to use function families as larger meaningful building blocks to recognize the structure of formulas and to graph formulas. The two ingredients, function families as larger building blocks and qualitative reasoning, are important thinking tools in the recognition of the structure of the formulas and so, in the reading of formulas, as they might relieve students' working memory. Our findings suggest that teaching graphing formulas to grade 11 students, based on recognition and qualitative reasoning, might be an efficient means to promote student insight into algebraic formulas in a meaningful and systematical way.

CHAPTER 5

The relation between graphing formulas by hand and students' symbol sense

This chapter is based on: Kop, P. M., Janssen, F. J., Drijvers, P. H., & van Driel, J. H. (2020b). The relation between graphing formulas by hand and students' symbol sense. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-020-09970-3

Abstract

Students in secondary school often struggle with symbol sense, that is, the general ability to deal with symbols and to recognize the structure of algebraic formulas. Fostering symbol sense is an educational challenge. In graphing formulas by hand, defined as graphing using recognition and reasoning without technology, many aspects of symbol sense come to play. In a previous study, we showed how graphing formulas by hand could be learned. The aim of the study we present here is to explore the relationship between students' graphing abilities and their symbol sense abilities while solving non-routine algebra tasks. A symbol sense test was administered to a group of 114 grade 12 students. The test consisted of eight graphing tasks and twelve non-routine algebra tasks, which could be solved by graphing and reasoning. Six students were asked to think aloud during the test. The findings show a strong positive correlation between the scores on the graphing tasks and the scores on the algebra tasks and the symbol sense used while solving these tasks. The thinking-aloud protocols suggest that the students who scored high on the graphing tasks used similar aspects of symbol sense in both the graphing and algebra tasks, that is, using combinations of recognizing function families and key features, and qualitative reasoning. As an implication for teaching practice, learning to graph formulas by hand might be an approach to promote students' symbol sense.

5.1 Introduction

Many students have serious cognitive problems with algebra, in particular with seeing structure and making sense of algebraic formulas with their abstract symbols (Arcavi et al., 2017; Drijvers et al., 2011; Kieran, 2006). The teaching of algebra often focuses on basic skills through practicing algebraic calculation in similar tasks (Arcavi et al., 2017). However, many students experience problems with when to use these basic skills and finding strategies to solve algebra problems: they lack symbol sense (Arcavi et al., 2017; Hoch & Dreyfus, 2005; 2010; Oehrtman et al., 2008; Thompson, 2013). Symbol sense concerns a very general notion of "when and how" to use symbols (Arcavi, 1994), and it functions as a compass when using basic skills (Drijvers et al., 2011). A lack of symbol sense leads to an over-reliance on basic skills, just learned methods, and on the symbolic representations, leading to poor achievements (Kieran, 2006; Knuth, 2000; Eisenberg & Dreyfus, 1994; Pierce & Stacey, 2007). However, it is not clear how to teach symbol sense appropriately (Arcavi, 2005; Hoch & Dreyfus, 2005). In a previous study, we showed how teaching graphing formulas by hand, defined as graphing using recognition and reasoning, without technology, to grade 11 students improved their insight into algebraic formulas (Kop, Janssen, Drijvers, & Van Driel, 2020a). Insight into algebraic formulas is an aspect of symbol sense and involves recognizing structure and key features of a formula, and qualitative reasoning with and about a formula. In the study presented here, we investigated whether graphing formulas by hand abilities are related to their symbol sense in a broader sense, that is, symbol sense while solving non-routine algebra tasks. Doing so, the study aims to contribute to our theoretical knowledge of students' symbol sense abilities and to inform teaching practice.

5.2 Theoretical background

The most important theoretical notion that guides this study is symbol sense. Fey (1990) was the first to mention symbol sense and described it as an informal skill required to deal effectively with symbolic expressions and algebraic operations. According to Fey, goals for teaching symbol sense would include at least the following basic themes: the ability to scan an algebraic expression to make rough estimates of the patterns that would emerge in numerical or graphical representation, to make comparisons of orders of magnitude for functions, and to inspect algebraic operations and predict the form of the result and judge the likelihood that it has been performed correctly. Arcavi (1994) elaborated on this idea and broadened the concept to all phases in the problem-solving cycle (Pierce & Stacey, 2004). According to Arcavi (ibid.), symbol sense would include:

- An understanding and a feel for the power of symbols, that is, understanding how and when symbols can be used in order to display relationships, generalizations, and proofs, and when to abandon symbols in favor of other approaches in order to make progress with a problem.
- An ability to manipulate and to "read" symbolic expressions as two complementary aspects of solving algebra problems.
- The awareness that one can successfully engineer symbolic relationships which express the verbal or graphical information needed to make progress in a problem, and the ability to engineer those expressions.

Drijvers et al. (2011) described symbol sense in relation to basic skills: symbol sense and basic skills are complementary. Basic skills involve procedural work with a local focus and an emphasis on algebraic calculations, whereas symbol sense involves strategic work, taking a global view on algebraic expressions/formulas and algebraic reasoning. A global view, or a Gestalt view, has to do with the ability to see an algebraic formula as a whole, to "read through" it, and to recognize its structure and global characteristics.

Related to the notion of symbol sense, Pierce and Stacey (2004) used the term algebraic insight to capture the symbol sense in transformational activities in the "solving" phase of the problem-solving cycle (from mathematical problem to mathematical solution) when using Computer Algebra Systems (CAS). Algebraic insight has to do with the recognition and identification of structure, objects, key features, dominant terms, and meanings of symbols and the ability to link representations (Kenney, 2008; Pierce & Stacey, 2004). Kenney (2008) used this framework of Pierce and Stacey for her research and added "know how and when to use symbols" and "know when to abandon a representation."

To develop symbol sense, graphing formulas by hand might be useful because it involves many aspects of symbol sense. First, graphing formulas is about linking the symbolic and graphical representation. Second, to efficiently graph formulas by hand, one has to recognize the structure and key features of a formula and to reason with and about formulas. Third, graphing a formula can be considered a visualization of a formula, and using such a visualization in problem-solving requires knowing about what is represented and where to look for. We will now elaborate on these aspects.

Linking representations, such as formulas and graphs, is important in learning about functions (Janvier, 1987; Leinhardt et al., 1990) and might be used to give meaning to algebraic formulas (Kieran, 2006). Duval (1999) used the term registers of representations to indicate that each representation (formula or graph) has its own specific means and processing for mathematical thinking. He distinguished two types of transformation: treatments, transformations in the same representation, and conversions, transformations from one representation to another, like from formulas into Cartesian graphs. Conversions are at the core of understanding mathematics, but many students have problems learning these conversions, as it requires a change of register and the recognition of the same represented object in different representations (Duval, 1999, 2006).

In efficiently graphing formulas by hand, many aspects of symbol sense are involved. Research in expertise in graphing formulas by hand shows that experts' strategies could be described with combinations of different levels of recognition and qualitative reasoning (Kop et al., 2015). For recognition, experts use a repertoire of function families with their characteristics and key graph features like zeroes and turning points. In qualitative reasoning, the focus is on the global shape of the graph, ignoring what is not relevant in the situation, and using global descriptions. Qualitative reasoning is often used by experts in complex problem situations that are difficult to look through in detail, like in physical models (Bredeweg & Forbus, 2003). In the domain of graphing formulas, experts tend to use qualitative reasoning to explore (parts of) the graph, for instance, infinity behavior, increasing/decreasing of functions, stronger/weaker components of a function, and in the composition of two sub-graphs, after decomposing a formula in two sub-functions. Graphing formulas by hand is related to covariational reasoning, which is about coordinating two covarying quantities while attending to how they change in relation to each other (Thompson, 2013; Carlson et al., 2002). This covariational reasoning is critical in supporting student learning of functions in secondary and undergraduate mathematics (Carlson et al., 2002; Oehrtman et al., 2008). Carlson et al. (2015) showed that students were not able to reason "as the value of x gets larger the value of y decreases, and as the value of x approaches 2, the value of y increases" when they had to link the formula of $f(x) = 1/(x-2)^2$ to its graph.

Visualizing formulas through graphs is used in problem-solving for understanding the problem situation, recording information, exploring, and monitoring and evaluating results (Polya, 1945; Stylianou & Silver, 2004). Stylianou and Silver (2004) compared experts and novices in solving algebra problems and showed that experts know how to use graphs in solving algebra problems. Experts "see" relevant relations visualized in the graph and can use the graph for visual and qualitative explorations. Although novices have some declarative knowledge, they lack the necessary procedural knowledge to construct visual representations of general functions and to explore the graphs they have constructed. Such exploration requires a global view of the whole graph and not just a local apprehension (Duval, 1999) and is only possible when one is very familiar with the function (Stylianou & Silver, 2004). This matched Eisenberg and Dreyfus' (1994) ideas about the need for a repertoire of basic functions that one should simultaneously "see" in a graph as one thinks of the algebraic formula.

In sum, graphing formulas involves many essential aspects of symbol sense to solve algebra problems, like visualizing a formula through a Cartesian graph, taking a global view to read through a formula and enable recognition of the structure of a formula and/or its key features, and qualitative reasoning. In the current study, therefore, we focus on the relation between symbol sense involved in graphing formulas by hand and the symbol sense to solve non-routine algebra tasks. Aspects of symbol sense, learned and used in the context of graphing formulas, might be used in a broader domain of algebra tasks. In this study, this broader domain is restricted to algebra tasks that can be solved with graphs and reasoning, so without the use of algebraic calculations.

5.2.1 Research questions

The theoretical perspective described in the previous section led to the following main research question:

How do grade 12 students' abilities to graph formulas by hand relate to their use of symbol sense while solving non-routine algebra tasks?

We formulated two sub-questions. We expected a relation between students' abilities to graph formulas by hand and their abilities to solve algebra tasks with symbol sense, because graphing formulas can be seen as a subset of algebra tasks, and in graphing formulas by hand many aspects of symbol sense are involved. This led to the first sub-question:

To what extent are students' graphing formulas by hand abilities positively correlated to their abilities to solve algebra tasks with symbol sense?

In graphing formulas by hand, several symbol sense aspects are involved, and we expected that students would be able to use these symbol sense aspects also in the context of solving algebra tasks. In addition, we expected that when one is able to graph formulas by hand, one would see more possibilities to use this strategy (making a graph). This led to the second sub-question:

How is students' symbol sense use in graphing formulas similar or different from their symbol sense use in solving non-routine algebra tasks?

5.3 Method

We first describe the context of the study, that is, the position of this study in a larger research project; next, we address the participants, the symbol sense test, the data, and the way the data were analyzed.

5.3.1 Context of the study

This study is part of a larger PhD research project about studying how symbol sense might be taught. In two previous studies, we analyzed expertise in graphing formulas by hand and identified main components of symbol sense used by the experts, that is, recognition of function families and key features from the structure of formulas, and qualitative reasoning. In a third study, a group of 21 students from the first author's school were taught how to graph formulas by hand, using recognition and reasoning, in a series of five lessons of 90 min. The series of lessons started with the recognition of basic function families with their characteristics. Students learned about transformations and about using qualitative reasoning by focusing on the global shape of the graphs and using global descriptions (e.g. "it is a root-graph reversed"). Then these basic function families were used as building blocks when graphing more complex functions, like in $y = 2x + 4/x$ and $y = 2x\sqrt{5 - x}$. Complex functions could be decomposed in two basic functions, which both could be graphed. Explicit attention was paid to the composition of the two sub-graphs through qualitative reasoning. In a subsequent task, the focus was on recognizing graph features, like zeroes and turning points. When recognition falls short, one can do strategic explorations of parts of the graphs. In a subsequent task, students learned how to use qualitative reasoning for determining, e.g., infinity behavior of a function and increasing/decreasing. In the Method section 4.2 (in chapter 4), more details are given about this intervention. Pre-test results of this third study showed that the students had a lot of trouble with graphing formulas by hand; post-test results showed an improvement of their abilities. The current study is the fourth study, which focused on the relation between symbol sense involved in graphing formulas by hand and in solving algebra tasks.

5.3.2 Instruments

The main instrument developed for this study was a test on students' competencies and symbol sense use when graphing formulas by hand and when solving non-routine algebra tasks. Two types of tasks were used: type A tasks, in which the link between formula and graph was explicitly indicated, and type B tasks with no reference to graphs in the text. Students were asked to explain their answers.

The test was constructed in three steps. First, we looked for tasks that were used in other studies and adjusted them to our situation. Second, a first draft of the test was discussed with a professor in mathematics education and an experienced teacher. They were asked whether the tasks fit the grade 12 curriculum and whether they thought the students should be able to solve these tasks. Third, using their feedback, the test was constructed with eight type A tasks and twelve type B tasks. All teachers of the students involved in the study indicated that they thought that these tasks were challenging but, according to the curriculum, should be doable.

In the type A tasks, we explicitly used the word "graph" and addressed different aspects of linking formulas to graphs. Some of these kinds of tasks have been used more often in research: working with parameters (Drijvers et al, 2011; Heid et al., 2013), reverse thinking (finding a formula with a graph) (Keller, 1994; Drijvers et al., 2011; Duval, 2006), and evaluating a (part of the) graph made with a graphic calculator. The test started with type B tasks, because type A tasks might suggest using graphs in the type B tasks.

Type B tasks could be solved with only graphs and reasoning, but no explicit links to graphs were given in the text. These tasks should give information about the students' symbol sense use while solving algebra tasks. Some tasks have been used by others in assessing students' algebraic competences: number of solutions (Heid et al., 2013), inequalities (Kenney, 2008; Tsamir & Bazzini, 2004), and reasoning about the function (Kenney, 2008; Pierce & Stacey, 2007). In Appendix 5.1, we give the symbol sense test. The internal consistency and reliability of both types of tasks was deemed acceptable, based on Cronbach's alpha on the type A tasks being 0.70 and on the type B tasks 0.72. Deleting any task hardly changed the Cronbach's alpha.

5.3.3 Participants

In this study 114 grade 12 students from six different schools throughout the Netherlands were involved. The students had 45 min to finish the pen-and-paper test. All students were enrolled in the Dutch math B course that prepares for university studies mathematics, physics, and engineering. In regular education in the Netherlands, students learn about linear, quadratic and exponential functions in grade 8 and 9. In grade 10, the graphic calculator is introduced and power, rational, logarithmic functions are studied. In grade 11 and 12, calculus topics such as derivatives and integrals are taught. Graphing formulas by hand, without technology, is not a specific subject in the Dutch curriculum: graphs are normally made with the graphic calculator. Therefore, we expected that many students would have difficulties to graph formulas by hand and that they would score low on the graphing tasks in the symbol sense test. To investigate the relation between graphing formulas abilities and the abilities to solve algebra tasks, a broad range of scores on the graphing tasks was needed. To ensure this range of graphing abilities and to investigate how the teaching of graphing formulas by hand would affect students symbol sense abilities, 21 students from the third study (who were taught how to graph formulas by hand) were involved in the current study. The teachers of the five schools that were involved in the study volunteered to participate and differed with respect to years of teaching experience.

5.3.4 Procedure

In February 2017, the symbol sense test was administered to the 114 students. For each student, all answers on the tasks were scored as correct (score=1), partly correct (score between 0 and 1), or incorrect (score=0). For each student, the sum of the scores on the type A tasks resulted in a TA-score, and the sum on type B tasks in a TB-score. In addition, for both type A and type B tasks, the students' strategies were encoded, as far as these could be recognized from the written material. We looked for symbol sense strategies, like recognition of key features, decomposition in sub-formulas, and reasoning, and for other strategies, like making calculations (derivatives, and/or points). In the type B tasks, we also registered whether making a graph or a relevant part of the graph was used, as this were considered symbol sense strategies. When, in these type B tasks, symbol sense strategies were used, a strategy-score of 1 was given. However, when calculations were made, the strategy-score was 0. The sum of these scores resulted in a StratTB-score for each student. Besides the StratTB-score, an effective strategy-score (EffStratTB-score) was also calculated, because using a symbol sense strategy did not guarantee a correct solution. When the symbol sense strategy resulted in a score of 0.5 or higher on a task, the effective strategy-score was 1. The sum resulted in an EffStratTB-score for each student.

The scores on the written test, TA-, TB-, StratTB-, and EffStratTB-scores, are considered to be related to the general math ability of the students, and the general math ability of each of the 114 students was rated by their own teacher on a scale from 1 to 10 (called Math rating). In a one-way independent Anova on the students' Math ratings, no significant differences between the six schools were found.

In addition to the strategy-scores from the written tests, we wanted a more detailed picture of the relation between the symbol sense use in the graphing tasks and in the algebra tasks (sub-question 2). As we expected that symbol sense involved in graphing formulas might be used in solving algebra tasks, we asked six students who belonged to the group of 25% highest scoring students on the graphing tasks (scores of 3 until 7.5 out of max 8) to think aloud during the test. Two of these students had very high Math ratings (T and K), two had more than average Math ratings (A and M), and two had average Math ratings (Y and I). As our aim is to teach symbol sense to all students, these six students were also involved in the teaching graphing formulas by hand. Thinking aloud is not expected to disturb thinking processes and should give reliable information about problem-solving activities (Ericsson, 2006). The thinking-aloud protocols were transcribed.

5.3.5 Data analysis

The first sub-question was about the relation between the TA-scores and the three scores on type B tasks (TB-scores, StratTB-scores, EffStratTB-scores). The assumptions of regression, independent errors, homoscedasticity, normally distributed errors, and multicollinearity were met (Field, 2012). Because of the small number of items, the scores on the type B tasks were not normally distributed; therefore, bias corrected and accelerated bootstrap 95% CIs are reported.

The Math ratings were related to the scores on the type B tasks. To explore the relation between TA-scores and scores on type B tasks, we first used regression with the type B scores as dependent variables and the TA-scores as independent variable. Then, the Math rating was added also as an independent variable, to explore the influence of the Math rating on the scores of the type B tasks.

To get a more detailed picture of the relation between the TA-scores and the scores on the type B tasks, the group of 114 students were divided into four quartile groups, based on their TA-score. The 25% students with the highest TA-score formed the quartile group Q4, the second 25% students the Q3 group, etc. The Q4 group included 16 of the 21 students involved in the teaching graphing formulas. All written type A tasks were analyzed on the main strategies, that is, the use of recognition/reasoning, making calculations, and "no answer at all" (blank). The written type B tasks were analyzed on the main strategies recognition/reasoning, making a graph, making calculations, and blank. For each task and each group (Q4, Q3, Q2, Q1), the relative frequencies of the main strategies and also the mean scores of the four groups on the tasks were calculated.

The thinking aloud protocols could detail the main strategies that were used to analyze the written tasks. To analyze the thinking aloud protocols, these were transcribed, and the transcripts were cut into idea units, fragments that contained crucial steps of explanations (Schwarz & Hershkowitz, 1999). These idea units were encoded using Drijvers et al.'s (2011) framework and descriptions of experts' strategies in graphing formulas (Kop et al., 2015). Drijvers et al.'s framework uses the following categories: taking global view, reasoning, and strategic work. To detail the symbol sense in the category global view, strategies involved in graphing formulas were used: recognition of function families, using knowledge of prototypical graphs and other characteristics of the function family, and recognition of key features, without (instantly) knowing other characteristics. The category strategic work was split up: considering one's strategy and monitoring, and abandoning a representation (e.g. making a graph). Also, signs of lack of symbol sense were encoded; e.g., when time consuming and error-prone algebraic calculations were used, while the problem could be solved with recognition and reasoning. This led to the following codebook for the type A and type B tasks, that is explained in Table 5.1.

Table 5.1 Encoding the thinking-aloud protocols

Code category	Code	Description
Recognition	R1	Recognizing a function family (families) and using prototypical graphs and/or other characteristics of the function family
	R2	Recognizing and using key graph feature(s) (e.g. a vertical asymptote, zeroes, etc.)
Reasoning	Q	(Qualitative) reasoning about e.g. parts of graph (infinity behavior, in/decreasing, positive/negative, etc.), that is, using global descriptions (e.g., “a square root translated to the right”), ignoring what is not relevant in the situation
Strategic work	S1	Considering one’s strategy and/or monitoring
	S2	Abandoning a representation (making a graph), or changing a formula
Calculation (as an indication of lack of symbol sense)	C	Calculating points, derivatives, manipulation(s) of formulas, while the problem could be solved with recognition and reasoning

We give three examples to illustrate the encoding

As a first example, we consider student T working on task 14 (type B); they considered their strategy (S1); recognized the zeroes from the structure of the formula (R2); makes a graph (S2); and used qualitative reasoning when y-values are described in terms of positive/negative (Q):

> Hmm, not nice to expand the brackets and to differentiate the function; but can it be done smarter? (S1); we can say that there will be a zero at 0, and when $14 - 2x = 0$, so, at 7 and at 4 (R2); what shape do we have? (S2); for large x it is positive multiply negative multiply negative, so positive; for a very negative number we get a negative outcome (Q) (followed by a correct graph; score 1; encoding R2,Q,S1,S2).

As a second example, student A was working on task 4 (task B); they started with calculations (C); monitored their strategy (S1); recognized a key feature (asymptote) (R2); and used qualitative reasoning about function behavior in the neighborhood of x=3 (Q):

> First expand the brackets $x^2 - 13x + 30 + 40/(x-3)$ (C) ; can this be larger than 70?; I’m going to try to find the turning point (S1); then see whether it is a parabola with a max or something like that, but there is also a broken function; let’s see whether it is a parabola (R1) and see whether turning point is beneath or above 70 and then 40…..?(S1) (tries to calculate (C)) No, this will not work (S1); I think I calculate some points (S1); there is a vertical asymptote at x=3 (R2); so, when $x - 3$ is very small then this part becomes very large and dominate the rest of the function (Q); …$x - 3$ can be infinity small and then the fraction will be very large and easily above 70 (Q) (score 1; encoding: R2,Q,S1,C)

In example 3, student Y was working on task 3 (task B); they used graphs (S2) and prototypical graphs (R1); and described a “reversed” prototypical graph (Q):

> 2^x is equation of e^x(R1), so goes above (sketches a graph (S2); 2^{-x} goes the other direction (Q), so, they have 1 point of intersection (two correct graphs; score 1; encoding R1,Q,S2)

The categories to describe symbol sense in the codebook show some similarities with the Pierce and Stacey’s (2004) algebraic expectation framework. However, because our focus is on reading through formulas and making sense of them, the manipulation of formulas and equivalence of formulas plays a minor role compared with the Pierce and Stacey’s framework. The encoding was used to qualitatively study similarities and differences between the symbol sense use in the graphing and algebra tasks of each student.

5.4 Results

First, Table 5.2 shows the correlation between the variables Math rating, TA-scores, TB-, StratTB-, and EffStratTB-scores. The TA-scores were strong correlated with the type B scores.

Table 5.2 Pearson’s correlation coefficients with 95% bias corrected and accelerated CIs

	TA-score	TB-score	StratTB-score	EffStratTB-score
Math rating	.324 [.147, .493]	.479 [.318, .623]	.305 [.115, .470]	.424 [.245, .574]
TAscore		.630 [.492, .756]	.514 [.372, .646]	.590 [.438, .719]
TBscore			.689 [.598, .767]	.921 [.888, .945]
StratTBscore				.708 [.619, .781]

All correlations are significant ($p < .001$).

Next, the Math rating was added as an independent variable in the regression model with TB-score as dependent variable and TA-score as independent variable and later also with the StratTB-score and the EffStratTB-score as dependent variables. This resulted in slightly higher correlation coefficients, .694, .543, and .639, respectively, than were found in Table 5.2. In Table 5.3 more detailed information is given about the linear models.

Table 5.3 Linear model of predictors of type B scores with 95% bias corrected and accelerated CIs

Dependent variable		*b*	*SE B*	Partial correlations	*β*	*p*
TB-scores	Constant	-1.40 [-3.00, .08]	.72			.054
	TA-score	.68 [.48, .90]	.09	.57	.53	$.000^{***}$
	Math rating	.49 [.24, .75]	.11	.38	.31	$.000^{***}$
StratTB-scores	Constant	1.52 [-.53, 4.08]	1.12			.180
	TA-score	.79 [.49, 1.14]	.15	.46	.46	$.000^{***}$
	Math rating	.32 [-.10, .66]	.18	.17	.15	.071
EffStratTB-scores	Constant	-1.40 [-2.65, .07]	.68			$.041^{*}$
	TA-score	.57 [.37, .80]	.09	.53	.51	$.000^{***}$
	Math rating	.36 [.13, .57]	.11	.31	.26	$.001^{**}$

$^{*}p< .05$, $^{**}p< .01$, $^{***}p< .001$.

5.4.1 Students' symbol sense in Type A and Type B tasks

Tables 5.4 and 5.5 show a more detailed picture of strategies of the Q4 and Q3 groups on the selection of type A and B tasks. The students in the groups Q1 and Q2 scored much lower on the use of symbol sense strategies than the Q3 and Q4 groups. Therefore, we only report about the Q3 and Q4 groups for a representative set of tasks. For type A tasks, we choose task 9 and 11 (working with parameters), task 15 (finding a formula), task 16 (graphing a formula), and task 19 (checking features of a graph). For Type B tasks, we choose task 2 and 3 (number of solutions), task 4 (y>70), task 5 (inequality), task 7 (y-values), task 14 (about maximum), and task 18 (reasoning from formula). Tables 5.4 and 5.5 show that students of group Q4, as expected, used more symbol sense strategies than those in group Q3. See Tables 5.4 and 5.5.

Table 5.4 Strategy use (in percentages) of group Q3 and group Q4 on a selection of type A tasks

Strategy	Task 9		Task 11		Task 15		Task 16		Task 19	
	Q3	Q4	Q3	Q4	Q3	Q4	Q3	Q4	Q3	Q4
Blank	36	12	25	8	71	35	68	35	57	15
Calculation	14	19	18	4	7	8	11	4	0	0
Recognition Reasoning	50	69	57	89	18	58	21	61	43	85

Table 5.5 Strategy use (in percentages) of group Q3 and group Q4 on a selection of type B tasks

Strategy	Task 2		Task 3		Task 4		Task 5		Task 8		Task 14		Task 18	
	Q3	Q4	Q3	Q4	Q3	Q4	Q3	Q4	Q3	Q4	Q3	Q4	Q3	Q4
Blank	25	12	39	31	14	12	7	0	14	8	46	23	79	31
Calculation	36	27	18	19	36	15	14	50	4	0	39	27	4	4
Making graph	25	46	25	46	0	0	11	12	0	0	4	27	0	0
Recognition Reasoning	14	15	18	4	50	73	68	39	82	92	11	23	18	65

In the other tasks that are not included in Table 5.5, also, the Q4-students used more symbol sense than the Q3-students, who at their turn did much better than the groups Q1 and Q2. The Q4-students also had higher mean scores than the Q3 group. Sometimes the differences in mean scores were very large, e.g. on task 1 (.57 vs .18), task 2 (.47 vs .13), task 3 (.49 vs .17), task 4 (.33 vs .16), task 7 (.84 vs .50), task 13 (.45 vs .22), task 14 (.30 vs .00). However, we found exceptions, namely, the tasks about inequalities (task 5 and 6). In these tasks the Q3-students used more often symbol sense strategies than the Q4-students, and in task 5, the inequality $x(x-1) > 4x$, the Q3-students scored higher than the Q4-students (mean scores .70 versus .57). In the discussion we discuss these findings about the inequalities.

To qualitatively back up the quantitative findings, the thinking-aloud protocols were analyzed according the codebook of Table 5.1. Results of these analyses are presented in Table 5.6. In the columns “symbol sense on graphing tasks” and “symbol sense in algebra tasks” the strategies that were predominantly used by a student are reported, that is, strategies used in more than 30% of the tasks. In Table 5.6, we used the following codes:
R1, recognition of function families; R2, recognition of key graph features; Q, qualitative reasoning; S1, considering a strategy and monitoring; S2, abandoning a representation (e.g. making a graph); C, calculation.

Table 5.6 Scores and strategy use of the six thinking-aloud students

Student	Total score on graphing tasks (max 8)	Symbol sense on graphing tasks (type A)	Total score on algebra tasks (max 12)	Symbol sense on algebra tasks (type B)
T	7.5	R1, R2, Q	7.7	R1, R2, Q, S1, S2
A	6.2	R1, R2, Q	7.0	R1, Q, C
Y	4.7	R1, R2, Q	3.5	R1, Q, S2
I	3.0	R2, Q, C	4.5	Q, C
K	5.0	R1, Q	8.0	R1, Q, S2
M	3.7	R1, R2, Q	4.5	Q, C

The results in Table 5.6 seem to confirm the findings of the quantitative analyses of Tables 5.2 and 5.3, showing a relation between the scores on the graphing and algebra tasks. Table 5.6 shows that students often used recognition and qualitative reasoning when working on both kinds of tasks. As expected, the S2-strategy (abandoning a representation) was used more often in the algebra tasks than in the graphing tasks. Apart from the S2-strategy, there was some relation between the strategies used in both types of tasks; only students A and M showed larger difference in strategy-use between both kinds of tasks, as they often started calculations working on the algebra tasks. In Appendix 5.2, for each student, illustrative transcripts with encodings plus samples of their written work are given. It shows that, in both tasks, the students often needed combinations of recognition, reasoning, and strategic work, to solve the tasks. However, using more strategies was not always an indication of proficiency. For instance, the high achieving student K was short in their reasoning using function families and qualitative reasoning.

5.5 Discussion and conclusion

In this study, we investigated how students' graphing by hand abilities might be related to symbol sense abilities to solve non-routine algebra tasks. We designed a symbol sense test with graphing problems (type A tasks) and other algebra tasks that could be solved with graphs and reasoning, without algebraic calculation (type B tasks).

With respect to the first sub-question about the relation between the graphing formulas abilities and the abilities to solve the algebra problems with symbol sense, we found that the students who scored better on the graphing tasks also scored higher on the algebra tasks. General math abilities might explain this relation. However, when Math rating was added as an independent variable, the explained variance of the scores on the algebra tasks hardly increased. This suggested a positive relationship between students' graphing abilities and their abilities to solve algebra tasks. A similar positive relationship was found between the scores on the graphing tasks and the symbol sense scores on the algebra tasks (StratTB- and EffStratTB-scores), indicating that students who scored higher on the graphing tasks used more and more effectively symbol sense strategies while solving the algebra tasks. This relation was confirmed by the analyses between the Q3 and Q4 groups in Tables 5.4 and 5.5, which showed that the Q4-students used more symbol sense strategies than the Q3-students.

The second sub-question was about similarities and differences between symbol sense use in the graphing and algebra tasks. In the analyses of the thinking aloud protocols, we found that the six students used often similar symbol sense strategies in both the graphing and algebra tasks. Students' approaches to solve the graphing tasks could be described through combinations of recognition function families and using prototypical graphs and characteristics, recognition of key features of the function, and qualitative reasoning. To these combinations, the strategy "abandoning a representation" (making a graph) was added, when working on the algebra tasks. The high-scoring students more often used "making a graph" and had a larger repertoire of symbol sense strategies, than the other students, who more often tried to use calculations, and had trouble to use combinations of strategies. The findings suggest that, besides “make a graph,” students often used similar strategies in the graphing and algebra tasks.

The study aimed to contribute to the knowledge of symbol sense and to the students’ symbol sense abilities in graphing formulas and in non-routine algebra tasks. With respect to the main research question on how students’ graphing by hand abilities might be related to their symbol sense use while solving non-routine algebra tasks, our findings suggest that students could use their symbol sense involved in graphing formulas, that is, a combination of recognition of function families and graph features from the structure of the function, qualitative reasoning, and strategic work, to solve algebra tasks.

5.5.1 Limitations

Before discussing these results in more detail, we acknowledge that the study, of course, also came with limitations. The algebra tasks in our test were restricted to problems, predominantly using the variables x and y, that could be solved with graphs and reasoning, without algebraic calculation. Another issue is the combination of graphing and algebra tasks in one test which might have given suggestions to use graphs in the type B tasks. In a future study, these issues could be addressed by omitting explicit graphing tasks, by also using other variables than x and y, and by adding some tasks that need some algebraic calculation. In this article, the focus was on the relation between graphing abilities and the symbol sense abilities to solve algebra tasks. A next step would be to set up a quasi-experimental study, in which a group of students were taught to graph formulas by hand, using a control group and a pre-test and post-test.

In discussing the findings, we note that the results of this study seem to confirm earlier research about the problems Dutch students have with algebra: students have problems graphing formulas by hand (Kop et al., 2020a) and with identifying and using the structure of algebraic expressions (Van Stiphout et al., 2013). Regular teaching of algebra does not seem to develop these aspects of symbol sense. Although only Dutch students were involved in this study, literature about symbol sense (Arcavi et al., 2017; Drijvers et al., 2011; Kieran, 2006, Arcavi, 1994; Ayalon et al., 2015; Hoch & Dreyfus, 2005, 2010; Oehrtman et al., 2008) and personal conversations with teachers and scholars from other countries suggest that grade 12 students abroad have similar problems with symbol sense.

A remarkable finding, described in the "Results" section, was that in task 5, the inequality, the Q3-students scored higher and did use more symbol sense strategies than the Q4-students, who more often tried calculations to solve this task. We wonder why the Q4-students did not use their graphing skills in this task, as we expect that they could easily graph both formulas. Although we know from literature that students may over rely on the symbolic representation even when graphs are more appropriate (Knuth, 2000; Eisenberg & Dreyfus, 1994; Kenney, 2008; Slavit, 1997), we assume that the inequality triggered previously learned associations that hinder later learned symbol sense, as was suggested by student Y (see Appendix 5.2).

5.5.2 Implications

The findings of the current study suggest a positive relationship between the ability to graph formulas by hand and to solve non-routine algebra tasks and showed similarities in the symbol sense used in both kinds of tasks. The contribution of this study is that it describes this symbol sense through combinations of recognition of function families, and key graph features from the structure of the formulas, qualitative reasoning, and strategic work, and that it suggests how this symbol sense might be taught to students.

Graphing formulas and covariational reasoning, in the context of formulas, are related as both have a focus on global (qualitative) graphs. In this study, it is explicitly shown how and what qualitative reasoning was used by students. The importance of qualitative reasoning and its omission in regular math education have been stressed by Goldenberg et al. (1992), Yerushalmy (1997), and Duval (2006). In their elaborations about covariational reasoning, Moore and Thompson (2015) have problematized what they called static shape thinking, that is seeing a graph-as-a-wire. However, our findings show that the students successfully used prototypical graphs of function families as building blocks in their reasoning, when working on the graphing tasks and using the strategy "making a graph" in the algebra tasks. The need for such repertoire of functions that can be instantly visualized by a graph has been stressed by many, for example, by Eisenberg and Dreyfus (1994), Stylianou and Silver (2004), and Duval (2006). In combination with qualitative reasoning, such repertoire might provide a knowledge base that is needed to enable students using visualizations to solve algebra problems ("making a graph" strategy). Visualizations are more than just making or perceiving graphs, it is about noticing and understanding the whole that is represented with its features, for which a solid knowledge base is needed (Duval, 2006).

Teaching symbol sense is not easy (Arcavi et al., 2017; Hoch & Dreyfus, 2005). Before we describe our suggestions about teaching symbol sense, we discuss extant approaches. Pierce and Stacey (2007) suggested highlighting the formula's structure and key features in classroom discussions when working with graphs. Friedlander and Arcavi (2012) focused on meaningful reading of algebraic formulas and formulated small tasks that focused on, e.g., qualitative thinking and global comprehension. Kindt (2011) gave many examples of productive practice in algebra. These activities are valuable and can be easily added to existing lessons, but often manipulations of formulas play a central role in these activities, and they lack a systematic and a step-by-step development. Our approach to teach symbol sense focuses on enabling students to make sense of formulas and to read through formulas. In literature, it has been suggested that giving meaning to formulas can be done via linking representations of functions and/or via realistic contexts (Kieran, 2006). However, except for linear and exponential functions, formulas often cannot be directly linked to realistic contexts. Therefore, we choose to link formulas to graphs through graphing formulas. To learn about functions, many have recommended to use technology to link representations (Kieran, 2006; Kieran & Drijvers, 2006; Heid et al., 2013). However, Goldenberg (1988) found that students established the connection between formula and graph more effectively when they drew graphs by hand than when they only performed computer graphing. Others have recognized the need for pen-and-paper activities when working with technology (Arcavi et al., 2017; Kieran & Drijvers, 2006). Therefore, we tried to promote students' symbol sense through graphing formulas by hand.

In this study, we found strong correlations between students' graphing by hand abilities and their abilities to solve algebra tasks and their use of symbol sense while solving non-routine algebra tasks. These correlations could not be accounted for by students' general math abilities. Symbol sense involved in graphing formulas includes combinations of recognition of function families and key features from the structure of formulas and (qualitative) reasoning, and it is a subset of symbol sense involved in solving non-routine algebra tasks. In the current study, 16 of the 21 students who were involved in the teaching of graphing formulas by hand belonged to the 25% highest scoring students on the graphing tasks (Q4 group), who used more symbol sense when solving algebra tasks than the other students. The six thinking-aloud students, all involved in the teaching and belonging to the Q4 group, showed that they were able to use their symbol sense and graphing formulas abilities to solve the non-routine algebra tasks, that is, combinations of recognition, qualitative reasoning, and strategic work. This suggests that teaching symbol sense in the domain of graphing formulas by hand might be an effective means to teach essential aspects of symbol sense involved in solving non-routine algebra tasks. In a previous study we showed how to teach graphing formulas by hand, using these essential aspects of symbol sense (Kop et al., 2020a).

The current study provides more insight in the relation between symbol sense involved in graphing formulas by hand and in solving non-routine algebra tasks, in what aspects of symbol sense students can use while solving non-routine algebra tasks, and how this symbol sense might be taught. However, more research is needed to investigate how educational practices might benefit from these insights. Also, lower grades of secondary school could be included in such research, investigating Ruthven’s suggestion to start algebra with graphing activities, instead of algebraic calculations (Ruthven, 1990), and our suggestion to learn about functions through a combination of graphing tools to explore functions and graphing by hand activities to foster students’ symbol sense.

Appendix 5.1 Symbol sense test

Tasks 9, 10, 11, 15, 16, 17, 19, and 20 are the graphing tasks (type A tasks).

1) Give the number of zeroes of the function $f(x) = x(x^2 + 2) - 12$

A. no zeroes B. one zero C. two zeroes D. three zeroes E. more than three zeroes

2) Give the number of solutions of the equation: $5\ln(x) = \frac{1}{2}x - 10$

3) Give the number of solutions of the equation: $2^x = 2^{-x} + 3$

4) Can the *y*-value of $y = -0.1(x-3)(x-10) + 40/(x-3)$ become larger than 70?

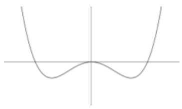
5) Solve the inequality: $x(x-1) > 4x$

6) Solve the inequality: $\frac{x^2-4}{x^2-9} < 0$

7) What outcome(s) can *y* have when $y = 24 - 0{,}01(x+5)^4$?

8) When *x* is very large, the function $f(x) = (3\mathrm{e}^{-x^2} + x^2)^3 + \frac{70}{x^4}$ can be approximated by: Choose the best alternative out of : A. $y = x^6$; B. $y = x^5$; C. $y = 70x^{-4}$; D. $y = 27\mathrm{e}^{-3x^2}$; E. none of these

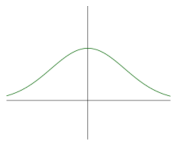
9) Here is a graph of $f_p(x) = x^4 - px^2$ for *p*=1.
Make a sketch of the graph of f_p for a value *p* >1.
Explain your answer.



10) Here is a graph of $f_p(x) = e^{-x^2+p}$ for *p*=1.

Make a sketch of the graph of f_p for a value *p*>1.

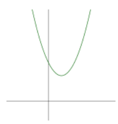
Explain your answer.



11) Here is a graph of $f_p(x) = (x-p)^2 + 2p$ for *p*=1.

Make a sketch of the graph of f_p for a value *p* >1.

Explain your answer.



12) Consider for each value of p the equation $2x^3 = px + 1$.

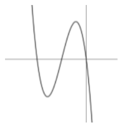
How many solutions can this equation have?

13) The number of different species of animals A in a domain can be modeled with the function $A = \frac{300}{2+3\cdot 0.87^t}$; *t*=0 is the year 2000. What does this formula tell about the number of different species in the domain?

14) Choose the correct alternative: A maximum of the function $y = x(14-2x)(8-2x)$ is situated in

A. [−4;0]; B: [0;4]; C: [4;7]; D: [7;14]

15) Find a formula that fits the graph.

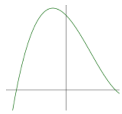


16) Make a sketch of the graph of $y = 4x\sqrt{x+5}$.

17) Make a sketch of the graph of $y = 3^x + 5x^{-2}$.

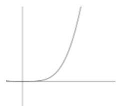
18) The formula $C = 0.13(1.92^{-t} - 1.92^{-6t})$ gives information about the concentration of medicine in mg/cm^3; *t* is the time in hours. What does this formula tell about the concentration *C*?

19) This is a part of the graph of $f(x) = (x^2-1)(x-\frac{3}{2})$.



Do you miss some characteristic features of this function?
If yes, graph the whole graph.

20) This is a part of the graph of $f(x) = \frac{20x^4}{x^4+2000}$



Do you miss some characteristic features of this function?
If yes, graph the whole graph.

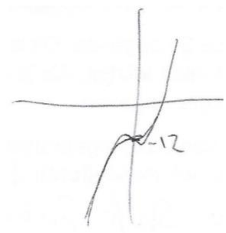
Appendix 5.2 Transcript of thinking aloud protocols with encodings plus samples from student work

To portray the students' symbol sense (or lack of symbol sense), we selected for each student representative fragments of their thinking aloud protocols about a certain task, combined with samples of their written work of that task.

Student T is a high-achieving student who used a broad repertoire of symbol sense strategies, including scanning and monitoring (S1); see example of task 14 in "Data analysis" section, and hardly used calculations.

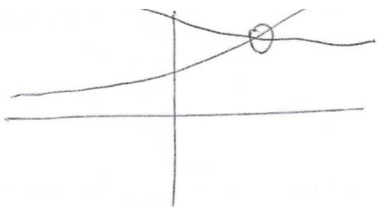
Student T working on task 1:

Hmmm, $x^2 + 2$ is just a normal parabola, with a minimum, 2 above (R1); multiplied by x; if we would take irrational numbers (complex numbers?), then we have 3 ; but when you multiply it (the parabola) with x, then right positive and left negative; then you would have only 1 zero (Q); it goes like this (gestured a prototypical x^3 graph (S2)), and that -12; I think there is only 1 zero (correct graph; score 1; encoding: R1,Q,S2)



Student T working on task 3:

2^x goes like this (S2, R1); 2^{-x}goes like this, and 3 higher (R1); so, 1 solution (score 1; encoding R1, S2)



Student T working on task 9:

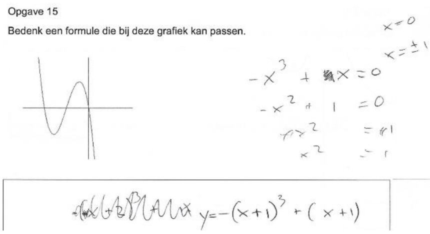
So, $x-1.5$; we have a zero, a zero at x=1.5 (R2); and zeroes at x^2 give zero at x=1; and ..; probably, this zero is at 1 and -1 (R2); then it goes further and is there a zero at 1.5, I guess; let's check what is happening at large x-values? y becomes positive (Q); so, (zero at) x=1.5 has to be added (sketches a correct graph; score 1; encoding R2,Q)

Student T working on task 12:

x^3 has this shape (sketches graph) (S2,R1); equals $px + 1$, which goes like this, or like this, or …(R1,Q); it can have 3 solutions, because it goes through (0,1); I think it can have up to 3 solutions (score 0.7; encoding R1,Q,S2)

Student T working on task 15:

A degree 3 and 2, and at x=0, it has to be 0; we have a x^3 (R1), but it has to be translated to the left (Q); this is the midpoint of x^3, and then it has to be negative: $-x^3$ (R2); then add a linear function, so that at x=0 it is 0 (checks at x=2 and considers a translation +8, but then focuses on the zeroes and calculates the zeroes, and translates the graph 1 to the left) (C) (score 1; encoding R1,R2,Q,C)



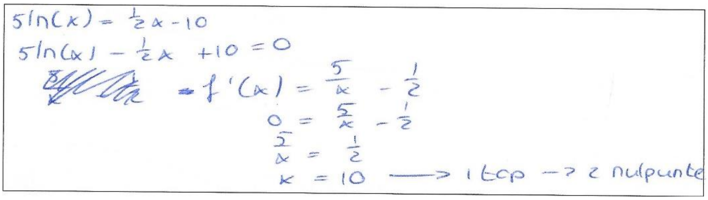
Student T working on task 18:

When t is increasing then this $(1{,}92^{-t})$ is becoming small (R2), and the other $(1{,}92^{-6t})$ even becomes faster small (R2, Q), because it is a negative number (in the power),…, does not matter, it just becomes very small and is decreasing (Q) (they does not pay attention to increasing part at the start; score 0.5; encoding: R2,Q)

Student A often started with calculation in the algebra tasks, but monitored their progress, and then used recognition and reasoning (see also example of task 4 in "Data analysis" section)

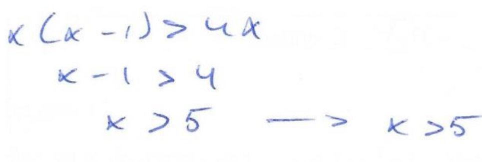
Student A working on task 2:

How can I find zeroes? (S1) I try to solve it. I think because it is not quadratic …; $5\ln(x) - \frac{1}{2}x + 10$ (S2)…;first calculate the derivative: $\frac{5}{x} - \frac{1}{2}$ (C) and searching for turning point; equals 0, so, x=10; there is a turning point at x=10 and when we substitute 10 then we get left 5…; we get two zeroes (writes x=10→1 turning point→two zeroes; score 0.7; encoding: S1,S2,C)



Student A working on task 5:

I think I first divide by x (C) because then it becomes much easier; so, $x - 1 > 4$; then it is very easy; so, $x > 5$ (score 0.3; encoding C)



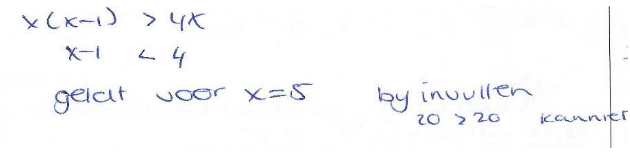
Student A working on task 15:

> I see two turning points and three zeroes; zero at x=0, and let's say, at x=−2 and x=−4; so, something like $x(x+2)(x+4)$ (R2); yes, then the turning points should be there; then taking care that when x is positive the formula-outcomes become negative; then we need $-x$ (R2); checks at x=−5 that y-value is positive) (S1) (correct formula $y = -x(x+2)(x+4)$; score 1; encoding R2,Q,S1)

Student Y is a hard-working student who thinks mathematics is difficult. They often used the strategy "make a graph" when solving algebra tasks (see example of task 3 in "Data analysis" section. Their work on inequalities suggested that previous learned procedures can give gave trouble (see task 5).

Student Y working on task 5:

> I have to think about inequality-sign; when dividing or multiplying by – or + it turns; but I don't remember this (S1); I think when dividing; but I'm not sure; I divide by x, so, $x - 1 <$ 4, that means it is true for x=5; to check: substitute 5 gives 20 (S1); > then larger or equal is not correct; when substituting, I get 20, but I do not know how to proceed (S1); I think the inequality sign reverses, but I'm not sure. (writes 20>20, not possible?; score 0; encoding: S1)

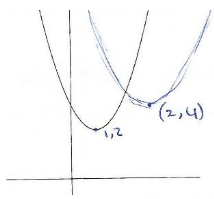


Student Y working on task 8:

> When x is very large then $70/x^4$ fades because it becomes very small, approaches 0 (Q); therefore alternative C is not correct; when x is very large then it becomes $y = x^6$; e^{-large} becomes almost 0, so, when substituting something very large in e^{-x^2} it approaches to 0 (Q); then only $(x^2)^3$is left; I doubt 5 or 6, but it is multiplying, so answer A (score 1; encoding: Q)

Student Y working on task 11:

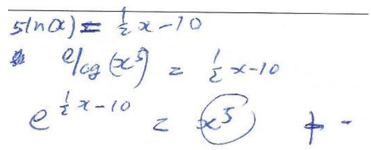
> (x−1) therefore a translation 1 to the right (R2); turning point is (1,2); when p is changing then it translates further; so, when we take p=2, then turning point is about here (R2) (sketches a correct parabola and give the coordinates of the turning point: (2,4); score 1; encoding: R2)



Student I thinks mathematics is very hard, but after the lessons about graphing formulas, student I developed more confidence in their mathematical thinking.

Student I working on task 2:

So, this is a long time ago. How do I do this?(S1) I transform this equation: $\log_e(x^5) = \frac{1}{2}x - 10$ (C)… Can I solve this equation? (S1) I can transform it into $x^5 = e^{\frac{1}{2}x-10}$ (C) but do I make any progress? … x^5 can only be positive or negative (R1). No, I do not know (score 0; encoding: R1,C,S1)

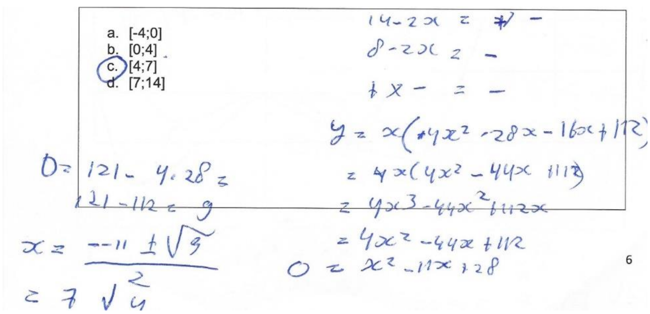


Student I working on task 7:

y cannot be larger than 24 because it is 'to the power 4' function (R1); then it is always positive, that is $0.01(x+5)^4$ is always positive (Q), so y cannot be larger than 24; so, $y \leq 24$ (score 1; encoding R1,Q)

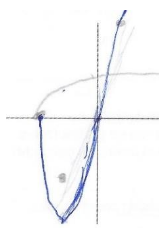
Student I working on task 14:

What happens when I make x very large: $14-2x$ negative, $8-2x$ negative, so, positive times negative is negative (Q); that you don't want; when x very negative, $14-2x$ very positive, times very positive, then ….;no; I expand brackets (of $(14-2x)(8-2x)$) (C), dividing by x; dividing by 4; (solves the equation $x^2 - 11x + 28 = 0$, finds x=4 and x=7)(C); so, turning point between 4 and 7 (score 0; encoding Q, C)



Student I working on task 16:

$4x$ goes like this (R1); $\sqrt{x}$ like this (R1); x+5 means that it starts at $x = -5$ (sketches $\sqrt{x+5}$) (R1); here it is 0 (points at x=0); here negative; I multiply these graphs; here it is positive (Q); I'm not sure (score 1; encoding R1, Q)



Student I working on task 19:

$(x^2 - 1)(x - 1.5)$ gives $x^3 - 1.5x^2 - x + 1.5$ (C); … turning point in view; no, when x is larger then x^3 larger but $-x^2$ larger, finally it will be negative (Q); so, all features in view: zeroes and y-values become negative when x is larger (score 0) (score 0; encoding: C)

Student K is a high-achieving student who often used their repertoire of function families and qualitative reasoning, and hardly used any calculations.

Student K working on task 2:

$\ln(x)$, so, e in the power something (gestures a correct graph) (R1,S2); $\frac{1}{2}x - 10$ runs like this (gesture) (R1); so, 2 solutions (two correct graphs, score 1; encoding R1,S2)

Student K working on task 10:

When p larger then….it is $e^p \cdot e^{-x^2}$(S2), so it is multiplied by larger factor (Q), multiplying relative to x-axis (R1) (sketches a correct graph; score 1; encoding R1,Q,S2)

Student M did not use their abilities to graph formulas to solve the algebra tasks; instead they often started with calculations.

Student M working on 3:

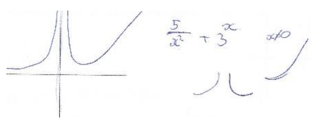
A "x" in the power; the +3 makes 2^x has to be larger than the 2^{-x}; …; it makes a difference whether x is positive or negative; look to the rules, with logarithm one gets …; if you use both 2^x and 2^{-x};….;dividing them; you get $2^{2x} = 3$(C), so one solution (not correct; score 0; encoding: C)

1 ~~an~~ oplossing
Als x $\frac{2^x}{2^{-x}}$ hebt zal dat zorgen voor 1
macht cijfer

Student M working on task 8:

this $(70/x^4)$ becomes very small so it faints (Q); so, only consider the first part; x^2 very large; $-x^2$ very negative, so, $1/e^{x^2}$ becomes very small because e^{1000} very large (Q) so, it will be x^2 in the power 3; it will be in the power 6 (score 1; encoding: Q).

Student M working on task 17:



(writes $5/x^2 + 3^x$) (S2); division, so x cannot be 0 (R2); …3^x ever increasing (R1); the other $(5/x^2)$ decreases to an asymptote (R2) and has only positive outcomes (Q)(sketches both sub-graphs); here, it is about 0+0 (Q) and then it becomes very large towards the y-axis; on the other side of y-axis, the closer to x=0

the larger y (Q)(sketches a correct graph; score 1;
encoding R1,R2,Q,S2)

CHAPTER 6

General Conclusions

6.1 Introduction

Students, even beyond secondary school, have cognitive and affective difficulties with algebra and its abstract symbols (Arcavi, 1994; Arcavi et al., 2017; Ayalon et al., 2015; Chazan & Yerushalmy, 2003; Drijvers et al., 2011; Kieran, 2006; Hoch & Dreyfus, 2005, 2010; Oehrtman et al., 2008). In regular algebra education, the focus is often on manipulations, starting with all kinds of basic skills, like expanding brackets, factorizing, calculating zeroes, extreme values, etc. However, many students do not know how to use these basic skills in solving algebraic problems, and find it hard to look through algebraic formulas and make sense of them: they lack symbol sense (Arcavi et al., 2017; Hoch & Dreyfus, 2005; 2010; Oehrtman et al., 2008; Thompson, 2013). Symbol sense concerns a very general notion of “when and how” to use symbols (Arcavi, 1994), and involves strategic work, taking a global view, and algebraic reasoning, whereas basic skills involves a local view, procedural working, and algebraic calculations. In this way, symbol sense functions as a compass when using basic skills (Drijvers et al., 2011). When students lack symbol sense, they have problems with giving meaning to and reading through formulas, resulting in a lack of confidence and in reluctance to engage in algebraic reasoning; so, students will focus on just learned methods in algebra lessons, in particular on basic skills, and on the symbolic representations (Arcavi et al., 2017; Kieran, 2006; Knuth, 2000; Eisenberg & Dreyfus, 1994; Pierce & Stacey, 2007). It is not clear how symbol sense can be taught effectively and efficiently in a systematical way (Arcavi, 2005; Hoch & Dreyfus, 2005).

In this research we investigated how to learn aspects of symbol sense, in particular reading through algebraic formulas and making sense of them, that is, to recognize structure and key features, and to reason with and about formulas. We called these aspects of symbol sense insight into algebraic formulas. Although identifying equivalent formulas is also an aspect of symbol sense, this aspect was not our first concern. Our research focused on grade 11 and 12 students, so students who in regular education already have learned about functions.

We chose to use graphing formulas with one variable by hand, so without technology, as a context to teach insight into formulas. In graphing formulas, all kinds of formulas can be involved and linking formulas to graphs can give students the opportunity to make sense of these formulas (Kieran, 2006; Radford, 2004). We chose to use graphing formulas *by hand* because the connection between formula and graph is more effectively established via graphing by hand than via computer graphing (Goldenberg, 1988). As our aim was students learning to read through formulas and to make sense of them, graphing formulas by hand does not here focus on a detailed graph in itself, but rather on making rough sketches of graphs.

To explore what knowledge and skills are needed to perform a complex skill like graphing formulas (complex because of the large variety of different formulas), it is recommended to study expert behavior (Kirschner & Van Merriënboer, 2008; Schoenfeld, 1978), as experts are supposed to use symbol sense when graphing formulas by hand. An analysis of expert behavior is crucial because guidelines for both what and how to teach on graphing formulas by hand can be partly derived from such an analysis. The overall research question in this thesis was: *How can teaching graphing formulas foster grade 11 and 12 students' insight into formulas and their symbol sense to solve non-routine algebraic problems?*

6.2 Results of partial studies

First, we present the main findings of the four separate studies of this thesis, followed by a discussion with implications, limitations and directions for future research.

6.2.1 Findings from study 1 (chapter 2)

In this study, we investigated experts' strategies in graphing formulas. Expertise literature indicates that problem solving could be described in terms of recognition and heuristic search. A two-dimensional framework with the dimensions recognition and heuristics was developed. The research questions addressed in this study were: Does the framework describe strategies in graphing formulas appropriately and discriminatively? Which strategies do experts use in formula-graphing tasks? In a case study, five experts and three teachers had to graph a more complex function ($y = 2x\sqrt{8-x} - 2x$) and had to find a formula that would fit a given graph, while thinking aloud. The protocols were transcribed and were cut into fragments which contained crucial steps of explanations.

The results show that all these steps from the protocols of all eight participants could be encoded within the two-dimensional framework. The solution process generated a path in the framework. Different strategies by the participants gave different paths in the framework. Therefore, we concluded that the framework was also discriminative. The experts used a range of strategies in graphing formulas. The main strategies seemed to be: recognizing function families and using their prototypical graphs, recognizing key graph features, using qualitative reasoning when composing two sub-graphs after decomposing a formula into two sub-formulas, and when exploring parts of the graph, e.g., infinity behavior. For recognition, a repertoire of basic functions (Eisenberg and Dreyfus, 1994) which can be instantly visualized by a graph is important. Expertise in graphing formulas does not involve calculations of derivatives, as all our experts seemed to hesitate to start such calculations and made mistakes when they did.

6.2.2 Findings from study 2 (chapter 3)

In the second study, we investigated experts' recognition in graphing formulas and addressed the research questions: Can we describe experts' repertoires of instant graphable formulas (IGFs) using categories of function families? What do experts attend to when linking formulas and graphs of IGFs, described in terms of prototype, attribute, and part-whole reasoning? IGFs can be seen as building blocks in thinking and reasoning with and about formulas and graphs. These building blocks can be combined (addition, multiplication, chaining, etc.) into new and more complex building blocks (e.g. IGFs $y = -x^4$ and $y = 6x^2$ combining to polynomial function $y = -x^4 + 6x^2$). Experts are expected to have more, and more complex, IGFs than novices, which generally enables them to graph formulas with fewer demands on working memory (Sweller, 1994).

The same five experts as in the first study worked on a card-sorting task to investigate what function families experts use, a matching task to investigate experts' recognition, and a thinking aloud multiple-choice task to portray experts' recognition processes. The experts' results in the card-sorting showed that the categories they constructed, and the category descriptions, were very similar, although some experts made more sub-categories (e.g., differences between parabolas with a max versus with a min). These descriptions were closely related to the basic function families that are taught in secondary school: linear functions, polynomial functions, exponential and logarithmic functions, broken functions, and power functions.

To analyze the thinking aloud protocols of the multiple-choice tasks, Barsalou's model of organized hierarchical knowledge with categories was used (Barselou, 1992). To portray students' concept image of functions, Schwarz and Hershkowitz (1999) used prototypicality (the use of prototypical members of a category or function family), attribute understanding (the ability to recognize attributes of a function across representations), and part-whole reasoning (the ability to recognize that different formulas or different graphs relate to the same entity). We combined both to formulate a Barsalou model for recognizing IGFs with function families, attributes and values, and graphs, to analyse how experts solved the multiple-choice task.

We found that experts' recognition of IGFs could be described with the Barselou model, in which function families, prototypes, a set of attributes and values of the attributes, and graphs are linked. For instance, given a logarithmic formula such as $y = \log_3(2x + 4)$, a prototype $y = \log_3(x)$ or $y = \log(x)$ was instantly identified and attribute reasoning (translation, domain $x > -2$, and/or vertical asymptote at $x = -2$) resulted in a graph. We also found that experts could easily work from a graph to a formula. For instance, a graph with attributes like domain $x > a$, a vertical asymptote at $x = a$ and concave down was instantly identified as a logarithmic function.

The findings show what knowledge experts used in recognizing IGFs: they used the basic functions to organize the function families, they used prototypes to handle other exemplars of function families, and also used prototypes and attributes to link graphs and formulas of function families. Our study suggests that only learning and practicing basic functions is not enough to become proficient in linking the formulas and graphs of functions. Students need to learn how to handle parameters in formulas and they need opportunities to integrate their knowledge of prototypes and attributes of function families into well-connected hierarchical mental networks. Through this study, we were able to adjust our Barsalou model based on Schwarz and Hershkowitz (1999) with linkages between attribute and value sets, prototypes and function families and with linkages from graph to attributes, prototypes, and function families. Our study gives an impression of an expert "library" of properties that may be helpful to further describe the recognition and identification of objects, forms, key features, and dominant terms used in Pierce and Stacey's algebraic insight (Pierce and Stacey, 2004; Kenney, 2008).

6.2.3 Findings from study 3 (chapter 4)

In the third study, we investigated how graphing formulas based on recognition and reasoning could be taught to grade 11 students and whether this graphing could improve students' insight into algebraic formulas. The research question addressed was: How can grade 11 students' insight into algebraic formulas be promoted through graphing formulas? The two-dimensional framework was the base for the design of an intervention consisting of a series of five lessons of 90 minutes. As graphing formulas can be considered as a complex task, a whole task approach, with support and reflection tasks, is recommended (Collins, 2006; Kirschner & Van Merriënboer, 2008; Merrill, 2013; Van Merriënboer et al., 2002). The importance of the meta-heuristic "questioning the formula" is stressed by Landa (1983), Arievitch and Haenen (2005), and Pierce and Stacey (2007). The five whole tasks reflected the levels of recognition in the two-dimensional framework. First, attention was paid to a repertoire of basic function families with their characteristics. Then to translations of the prototypes of the function families. In the third whole task, students practiced decomposing a formula into two instantly graphable sub-formulas, graphing the sub-formulas, and composing the sub-graphs. In a subsequent whole task, the focus was on the recognition of graph features from a formula, e.g., the zeroes and extreme values. In the last whole task, students explicitly practiced to reason qualitatively about infinity behavior, weaker and stronger components of a formula, in- and decreasing of functions, etc. This way of reasoning is often used by experts and is characterized by its focus on the global shape of the graph and global descriptions and ignoring what is not relevant.

The 21 grade 11 students from the first author's school who participated in the intervention made a written pre-, post-, and retention test after four months, which contained a graphing task and a matching task that was similar to the one used in study 2. Six students were asked to think aloud during the graphing tasks in the pre- and post-test. We found that in the pre-test the students lacked insight into formulas, and the thinking-aloud protocols suggested a lack of recognition and reasoning skills. The post-test results showed that students had improved their recognition of function families and graph features and their qualitative reasoning abilities. The students themselves indicated that their recognition and performances in graphing formulas had improved and that they understood formulas better. We interpreted that as the students having improved their insight into formulas. In the retention test, the scores on the graphing task and multiple-choice task were, as expected, lower than in the post-test, but higher than in the pre-test. This suggests a long-lasting effect of the intervention.

The findings of this study suggested that, although many students still had problems with more complex formulas, teaching graphing formulas to grade 11 students, based on recognition and qualitative reasoning, might be a means to promote student insight into algebraic formulas in a systematical way.

6.2.4 Findings from study 4 (chapter 5)

In study 4, the research question was: How do grade 12 students' abilities to graph formulas by hand relate to their use of symbol sense while solving non-routine algebra tasks? We used the sub-questions: (1) To what extent are students' graphing formulas by hand abilities positively correlated to their abilities to solve algebra tasks with symbol sense? And (2) How is students' symbol sense use in graphing formulas similar or different from their symbol sense use in solving non-routine algebra tasks?

The 21 students who were involved in the intervention in the third study made a written symbol sense test, together with 91 grade 12 students from five different schools throughout the Netherlands. The test consisted of eight graphing tasks and twelve non-routine algebra tasks, which could be solved by graphing and reasoning. We determined from the written test whether students could solve the tasks and what strategies they used (using symbol sense strategies like graphing or reasoning, or non-symbol sense strategy like calculations). Six students who participated in the intervention were asked to think aloud during the test.

With respect to the first sub-question, we found a positive correlation between students' graphing abilities and their abilities to solve algebra tasks and their symbol use when solving these tasks, also when corrected for students' general math abilities. High scoring students more often used strategies like making a graph and reasoning, and less often started calculations than students who were less successful.

With respect to the second sub-question, we found that 16 of the 21 students who were involved in the teaching of graphing formulas by hand in the intervention of study 3, belonged to the 25% highest scoring students on the graphing tasks. These students used more symbol sense when solving non-routine algebra tasks than the other students. Among these 16 students were the six thinking-aloud students, who showed that they used similar aspects of symbol sense in both the graphing tasks and the algebra tasks, including combinations of recognition function families and key graph features and qualitative reasoning. As symbol sense involved in graphing formulas is a subset of symbol sense involved in solving non-routine algebra tasks, these findings seem to confirm our expectations that students who are able to graph formulas by hand can use these abilities in a broader domain of non-routine algebra tasks. This suggests that teaching to graph formulas by hand might be an approach to promote students' symbol sense to solve non-routine algebraic problems.

6.3 Discussion and Conclusion

The aim of this research was to promote aspects of students' symbol sense, that is, students' abilities to read through formulas, to make sense of formulas, and to use this symbol sense when solving algebraic problems. The overall research question that led the research was: *How can teaching graphing formulas foster grade 11 and 12 students' insight into formulas and their symbol sense to solve non-routine algebraic problems?*

To answer this question, we assumed that it is extremely important that students can make sense of their algebra activities, and that students need some flexibility in their algebraic reasoning. Algebraic problems are not always represented in such way that students can instantly use their basic skills, and even if this is the case, they have to be able to recognize and select correct basic skill(s). If students cannot make sense of their algebra activities, they will not develop confidence in their algebraic reasoning, which result in a reluctancy to engage in algebraic reasoning, leading to inflexibility. Therefore, we chose a small but rich domain in algebra, namely graphing formulas. In graphing formulas, many different kinds of algebraic formulas are involved, it requires students to read through formulas, and it allows students to make sense of formulas. As our aim was to foster insight into formulas, we restricted the tasks to interpreting formulas and ignored algebraic manipulations, which are often at the core of regular algebra education. These restrictions would allow students to learn expertise in such a small domain and to make sense of algebraic formulas. We chose to graph formulas *by hand* because connections between formula and graph established via by hand activities seem to be more effective than via computer graphing. As experts are supposed to use insight into formulas, we investigated expert behavior in graphing formulas by hand and detected essential thinking processes. We described expert thinking in terms of recognition and reasoning in a two-dimensional framework. This gave us a clue about what to teach: a repertoire of function families with their characteristics and prototypical graphs, recognizing key graph features, and qualitative reasoning. We designed an intervention of five lessons of 90 minutes based on the two-dimensional framework: the GQR-design (Graphing based on Qualitative reasoning and Recognition). Through whole tasks, with help and reflection questions, and using "questioning the formula" as a leading meta-heuristic, graphing formulas was taught step-by-step and in a systematical way. In this GQR-design, explicit attention is paid to the interplay between recognition and reasoning by using combinations of function families with their prototypical graphs as building blocks, key graph features, and qualitative reasoning. The whole tasks approach forces students to take a global view for recognition, to reason and argue, and to consider their strategies, which are essential aspects of symbol sense (Drijvers et al., 2011). We expected that students could use these aspects of symbol sense learned through graphing formulas while solving non-routine algebra tasks. We designed a symbol sense test with non-routine algebra tasks that could be solved via recognition, reasoning, and making a graph. Results in a symbol sense test suggest that the students involved in the intervention were able to use their symbol sense in graphing formulas and were able to use graphs as visualizations while solving the non-routine algebra tasks. We conclude that teaching graphing formulas by hand with our GQR-design could be an effective means to teach students in the higher grades of secondary school aspects of symbol sense, like insight into algebraic formulas, that can be used to solve non-routine algebra tasks.

6.3.1 Contributions to Theory and Practice

The contributions of this research to the knowledge about symbol sense and teaching symbol sense are (1) that it describes the nature of expertise in terms of recognition, reasoning and its interplay, and shows how this can be elaborated for the domain of graphing formulas, and (2) that it shows how grade 11 students can acquire insight into algebraic formulas through an innovative intervention about graphing formulas, and (3) that it explores how symbol sense might be taught to students. We elaborate these three aspects.

As a first contribution of this research, we described the nature of expertise in terms of recognition and heuristics. From expertise research, it is known that experts have more structured knowledge compared to novices. This enables them to recognize more and make more sophisticated problem representations, which allow for more efficient searching in a problem space (Chi et al., 1981; Chi et al., 1982; Chi, 2011; De Groot, 1965; De Groot et al., 1996; Gobet, 1998). The level of recognition determines the problem space and, as a consequence, the heuristic search: recognition guides heuristic search. Based on this, we identified a two-dimensional framework to describe strategies in graphing formulas with different levels of recognition and heuristics, like qualitative reasoning about, e.g., infinity behavior, weaker/stronger components of a formula, etc. This two-dimensional framework stresses the interplay between recognition and reasoning. This approach differs from, for instance, descriptions of knowledge bases, in which mathematical competences are described in terms of conceptual and procedural knowledge, together with strategic competence, adaptive reasoning, and productive disposition (Kilpatrick, Swafford, & Findell, 2001). Conceptual knowledge refers to knowledge of concepts including principles and definitions which are connected in a network, and procedural knowledge refers to knowledge of procedures, including action sequences and algorithms used in problem solving (Star & Stylianides, 2013). The integration of the five different strands of mathematical competence has been stressed (Kilpatrick, Swafford, & Findell, 2001), but to our knowledge, has not led to models in which these components are actually integrated. The contribution of this research is that it shows how expertise in graphing formulas could be described through an interplay between recognition and domain-specific heuristic search. Recognition can be related to conceptual knowledge about function families with their characteristics and graph features. Domain-specific heuristics in graphing formulas like qualitative reasoning when composing two sub-graphs (after decomposing a formula into sub-formulas) and when exploring infinity behaviour, weaker/stronger components of a formula can be related to procedural knowledge. Also, strategic competence and adaptive reasoning can be related to the two-dimensional framework. The strategic component can be related to different routes in the framework that might lead to the graph of a formula, so to different strategies. Adaptive reasoning is included in the framework because, on each level of recognition, the framework gives suggestions to make progress in the graphing.

Describing expertise in terms of recognition and heuristics in a two-dimensional framework, also seems possible in other domains of algebra. Pouwelse, Janssen and Kop (submitted) proposed a framework with recognition and heuristics for finding indefinite integrals in calculus. The framework could be used as an instrument for designing teaching material but also as an instrument in teacher professional development, as it might allow teachers to reflect on their current teaching and inspire them to adjust it. Further research is needed to explore how the interplay between recognition and heuristic search in other domains could be described and used in designing teaching and/or in teacher professional development.

As a second contribution of the research, we showed how grade 11 students can acquire insight into algebraic formulas through graphing formulas. We used the two-dimensional framework as a base for our GQR-design (Graphing based on Qualitative reasoning and Recognition), an innovative series of lessons on graphing formulas. The levels of recognition in the framework were used as the meta-heuristic “questioning the formula”, stimulating students to take a global view before starting their graphing work. These levels of recognition were also used to formulate five whole tasks, with help and reflection questions. Our GQR-design is an innovative approach to teach about functions in a systematical and structural way in grade 11, but also in grade 12. Our research focused on students in the higher grades of secondary school, who learned in grade 8 and 9 about basic functions, like linear, quadratic, exponential functions, and in grade 10, with using graphic calculators, about power, rational, logarithmic functions. Much research is known about learning linear and exponential functions in lower secondary school, and how students might make sense of these functions and acquire insight into the formulas by linking them to realistic contexts. However, in higher secondary school, students must deal with many more different functions, which cannot easily be linked to realistic contexts.

The GQR-design differs from both regular and other innovative approaches to learning about functions, in particular regarding the link between formulas and graphs. Our approach differs from regular education about functions that often focuses on the manipulation of algebraic expressions (Arcavi et al., 2017; Schwartz & Yerushalmy, 1992) and on using graphing tools to explore function families and to work on calculus problems. In comparison to regular approaches, in the GQR-design, explicit attention is paid to recognition of function families and key graph features and to reasoning with and about functions. The first two whole tasks focus on a repertoire of function families with their characteristics, which are used as building blocks of formulas in the other whole tasks. In the fourth whole task, students learn to read key features from the formulas, e.g., reading the zeroes or extreme values from a formula, and in each whole task, attention is paid to reasoning, for instance about parameters of function families (in the second whole task), when composing two sub-graphs (in the third whole task), when exploring parts of the graph (e.g., about infinity behavior of the function in the fifth whole task).

Other innovative approaches often focus on reasoning about functions, using graphing tools; for instance, about the composition and translation of graphs (Schwartz & Yerushalmy, 1992; Yerushalmy & Gafni, 1992; Yerushalmy, 1997), about the role of parameters (Drijvers, 2003; Heid et al., 2013), and about special function families (Heid et al., 2013). In comparison to our approach, these approaches do not explicitly pay attention to qualitative reasoning and to the recognition of function families. Our GQR-design focuses on qualitative reasoning, with its focus on the global shape and on global descriptions, and with ignoring what is not relevant in the problem situation. The importance of qualitative reasoning and its omission in a mathematics curriculum was already stressed by Leinhardt et al. (1990), Goldenberg et al. (1992), Yerushalmy (1997), and Duval (2006), but to our knowledge, this idea has never been implemented in concrete and systematic teaching approaches. In our GQR-design, we use basic function families as building blocks for formulas, which, according to Davis (1983) could help students to recognize the structure of a formula. Davis (1983) has suggested that students learn to use larger thinking units, because they often work on an atomic level, that is, the role of each number and variable is analyzed, which makes it difficult to recognize any structure. (Davis, 1983). The larger thinking units and the use of qualitative reasoning might relieve working memory (Sweller et al., 2019), and might account for the results on the pre-, post-, and retention tests in study 3.

The third contribution of this research is that it shows how symbol sense to solve non-routine algebra tasks might be taught to students. Symbol sense is difficult to teach (Arcavi et al., 2017; Hoch & Dreyfus, 2005), probably because symbol sense is a very broad concept, involved in many aspects of algebraic thinking and working. Therefore, it seems hard to teach symbol sense in a systematical way, and consequently, students have problems with symbol sense. Students, also in upper secondary school, seem to avoid engaging in algebraic thinking and reasoning, and to focus on just learned methods (Arcavi et al., 2017; Kieran, 2006; Knuth, 2000; Eisenberg & Dreyfus, 1994; Pierce & Stacey, 2007). In regular education, many teachers and students focus on basic skills and manipulating formulas and expressions (Arcavi et al., 2017), expecting that students will develop symbol sense through this kind of practice. Innovative approaches focus more on reasoning, and give suggestions how to teach this, for instance, through using productive practices, such as reverse thinking and constructing examples (Friedlander & Arcavi, 2012; Kindt, 2011), using rich, collaborative tasks (Swan, 2008), and snapshots for classroom discussions (Pierce & Stacey, 2007).

At the core of the current research is the idea of teaching symbol sense in a small domain of algebra, graphing formulas by hand, allowing students to develop expertise in this domain. Graphing formulas can be considered a small domain in algebra because the task is clear and easily recognizable (make a sketch of the graph), but it is also a complex task because of the many different (types of) formulas that can be involved. The teaching of graphing formulas should focus on essential aspects of symbol sense, among them taking a global view for recognition, (qualitative) reasoning, and strategic work, which would allow some transfer of these essential aspects of symbol sense to a broader domain of algebra. In the GQR-design, students can learn these essential aspects of symbol sense in a systematical way. They learn how to use recognition, reasoning, and the interplay between recognition and reasoning, with thinking tools like the meta-heuristic "questioning the formula", a repertoire of function families, and qualitative reasoning. The GQR-design differs from other approaches by explicitly teaching these thinking tools, which are often implicit in other approaches. Our research shows that students obtained insight into formulas, and learned essential aspects of symbol sense, which they could later use while solving non-routine algebraic problems in the symbol sense test. The students involved in the intervention indicated that they thought they understood functions better, could visualize formulas better, in particular basic functions, and indicated that qualitative reasoning was very new and motivating for them ("we now use global reasoning; it is fun, this kind of reasoning"). This suggests that the GQR-design is a motivating and systematical way to teach students aspects of symbol sense.

6.3.2 Limitations and suggestions for future research

In this section, we address the limitations of the different studies and suggest directions for further research. In study 1 and 2, only five experts participated: two mathematicians who had been teaching calculus and analysis to first-year students at university, an author of a mathematics textbooks, who was also a teacher in secondary school, a math teacher who was involved in the National Math Exams and was a secondary school teacher, and a math teacher educator in university. All had a master's degree in mathematics, and two had a PhD in mathematics and had been working as a teacher for more than 20 years. During their career, they had been graphing many formulas without technology. Therefore, we considered them experts in graphing formulas, since we did not know other criteria for expertise in this domain of graphing formulas. We realized that this criterion for expertise was a bit vague. Testing a larger group of potential experts before describing expertise in graphing formulas might give another, more detailed picture of expertise. The experts worked on only two tasks, due to the labor-intensive method for strategy assessment: graphing one complex formula ($y = 2x\sqrt{8 - x} - 2x$) and finding a formula fitting a given graph. Although we expected that most common strategies were captured in the two-dimensional framework, future research, involving more and other functions could provide information on whether alternative strategies not mentioned in the framework are used regularly.

In study 3, we investigated the GQR-design, a series of lessons on graphing formulas by hand, that is based on the two-dimensional framework and focuses on teaching expert strategies in graphing formulas, that is, a combination of recognition and qualitative reasoning. We used the theory of teaching complex skills to formulate three design principles: the use of whole tasks, to support students when working on these whole tasks, and the use of the meta-heuristic “questioning the formula”. The levels of recognition of the two-dimensional framework form the backbone of the series of lessons, as they reflect the five whole tasks and the meta-heuristic “questioning the formula”. The GQR-design is meant for higher grades in secondary school, when students already have learned about basic function families, about transformations, and graph features. Thus, this series has a formative character. Therefore, ideas of Swan and Burkhardt about formative assessment were used (Swan, 2005; Burkhardt & Swan, 2013), resulting in whole tasks about differences and similarities of two graphs or formulas (whole task 2) and about categorizing functions according to their infinity behavior (whole task 5). Through whole tasks, students are confronted from the start with the full complexity of graphing formulas, that is, the interplay between recognition and reasoning. Because of time constraints, on each level of recognition, only one whole task was used. Although the limited time demands of this series is a strong point, we would recommend to consider Kirschner and Van Merriënboer’s (2008) suggestion to use more variability in the whole tasks (so, more whole tasks on each level of recognition), with more practice of the integration and coordination of all sub-skills. A second design principle was to support students when working on the whole tasks. For each whole task, help is offered in the teaching material, as well as reflection questions in which own examples are demanded. Other aspects of support were students cooperating in pairs or groups of three, and the modeling of expert behavior in graphing formulas by the teacher. A suggestion might be to use video to show the modeling of expert thinking processes in graphing formulas. To improve students’ reflection, one might consider the implementation of cumulative reflection tasks, which require students to reflect not only on the just completed task but also on all previous tasks. The third design principle was using the meta-heuristic “questioning the formula”. The students improved their recognition, as was shown in the post-test and retention test, but the thinking aloud protocols did not show that students had started to consciously question the formula. This might mean that the students had already automatized the habit of questioning the formula, as was our purpose. However, because only some of the better performing students showed that they considered their strategies, we believe that more attention should be paid to the habit of consciously questioning the formula.

Several aspects in the series of lessons might be adjusted when the series of lessons is used a next time. Although we thought that transformations of basic functions should be familiar to the students, the whole task on transformations (whole task 2) took more time and was more difficult for them than we had expected. We suggest taking more time for this whole task. Explicit use of qualitative reasoning was new to the students, and this kind of reasoning was demonstrated several times by the teacher. The results of the pre- and post-test suggested that the students had started to use qualitative reasoning, but many of them still had problems with using qualitative reasoning to compose sub-graphs and to explore parts of a graph. We suggest paying more attention to this qualitative reasoning, in particular in whole task 3 and 5. Another point for consideration is to pay more attention to third- and fourth-degree polynomials as function families. Probably because polynomial functions were not explicitly considered as a function family in the teaching material and because of practicing to decompose a formula into sub-formulas, many students used decomposing $y = -x^4 + 2x^2$ into $y = -x^4$ and $y = 2x^2$, but then had problems with the composition of the two sub-graphs. Recognizing $y = -x^4 + 2x^2$ as a member of the fourth degree polynomial function family with its characteristics would be helpful. In such situation, one might consider factorizing the formula ($y = x^2(-x^2 + 2)$), which would enable one to easily find the zeroes of the function. These findings also suggest that small manipulations, like factorizing might be helpful and needed, and we suggest to include these in a next series of lessons.

In the intervention, only one class from the Netherlands was involved, and no comparison group was included. However, one year and two years later, the same series of lessons from the intervention was used in two other groups in the same school, both of 23 students. Both groups made the same post-test that was used in study 3. The scores of both groups showed similar results to those of our 21 students in the study 3. Although this might be an indication that students can develop insight into formulas via this series of lessons, we suggest future research including more students and teachers to further investigate whether and how students can improve their insight into formulas through GQR-design.

In study 4, a symbol sense test was designed to investigate whether grade 12 students' abilities to graph formulas by hand were related to their abilities to solve non-routine algebra tasks and to their use of symbol sense. These algebra tasks were limited to those that could be solved using graphing and reasoning (e.g., discussing the number of solutions of a given equation or the *y*-values of a function), as our research focused on reading through formulas and making sense of them, and not on algebraic calculations. In the symbol sense test, we used a combination of graphing tasks and algebra tasks. Such a combination in a single test might suggest using graphs when working on the algebra tasks. As "making a graph" is considered a symbol sense strategy, we suggest being careful with explicit graphing tasks in future symbol sense tests. In the current test, the variables *x* and *y* were used predominantly to make the test more recognizable for the students. In future studies, we suggest using other variables than *x* and *y* more often, because working with such variables is also an aspect of symbol sense. Another issue is the selection of non-routine algebra tasks in the symbol sense test. In the current symbol sense test, we had several types of tasks: about the number of solutions of equations, about the *y*-values of functions, about inequalities, about approximations of functions when *x* is very large, about what information a formulas tell about a give situation, about the location of a maximum of functions. In future tests, we want to broaden the scope of these non-routine algebra tasks that can be solved with combinations of recognition and reasoning, so without algebraic calculations, like tasks about integrals and graph features, for example, "calculate $\int_{-4}^{4} x^3 e^{-x^2}\, dx$", and "how many zeroes, extreme values, and points of inflection has $y = (x-3)^2(x-5)^2$". As indicated above, the tasks in the current test could be solved with reasoning and graphing, and algebraic calculations were not needed. However, algebraic problems often require a combination of reasoning, graphing *and* calculations. Solving algebraic problems with symbol sense includes recognizing when reasoning is sufficient and when calculations are required to solve the problem. A next step is also to include tasks in which one has to consider whether calculations are required or not. Two examples to illustrate these kinds of tasks. First, "how many solutions has the equation $3.6(1 - \mathrm{e}^{-2.5t}) = 10t$?". Second, "consider the quadratic function $y = x^2 - 3$ and a family of linear functions $y = ax + 3$; for which value of a is the area bounded by parabola and by the line minimal?" (Stylianou & Silver, 2004).

In the symbol sense test, we found that students had problems with solving inequalities like $x(x-1) > 4x$. The 25% best graphing students were less successful on this task (score of .57) than the second 25% best graphing students (score of .70). Instead of using their graphing abilities, half of the best graphing students started calculations and were often unsuccessful. These findings seem to suggest that an inequality triggered previously learned associations, and that such associations might hinder later learned symbol sense. Further research is needed to investigate how just learned symbol sense can be incorporated in students' strategies and habits to deal with algebraic problems.

In study 4, we suggested that graphing formulas based on recognition and reasoning might be a means to teach symbol sense in grade 11 that could be used by students to solve non-routine algebra tasks. More research is needed to clarify this suggestion. A next step might be to set up a quasi-experimental study, in which a group of students is taught to graph formulas like in the intervention, using a control group and a pre-test and post-test. As we expect that difficulties with insight into algebraic formulas and symbol sense are not exclusive to Dutch students, students and teachers from other countries should also be included in future studies.

The ability to read through formulas and make sense of them is an important aspect of symbol sense and will remain important in the future. We expect that technology will take over many algebraic manipulations. However, to be able to use this technology, people have to interpret results, make global estimations about results, and understand what is going on. For this purpose, they will need some symbol sense, have to be able to question the problem and to read through formulas in models, to use visualizations in problem solving, and to have confidence in their own algebraic reasoning. Therefore, students in school need to develop some formula sense, that includes:

- making sense of a formula
- using function families as building blocks of formulas
- identifying and using the structure of a formula
- interpreting the role of parameters in a formula
- ignoring what is not relevant for a problem situation
- using a graph as a visualization of a function
- reasoning with and about formulas

These ideas might be interesting for developing curricula for secondary school. In many curricula the importance of symbol sense is acknowledged (e.g., NCTM, 2000). However, this is often in terms of understanding and not in concrete terms, such as in our list of formula sense above. In this research, we showed how reading through formulas and making sense of them can be taught to students via our GQR-design and that these aspects of symbol sense can be used by students when solving non-routine algebra tasks. We suggest that a similar approach might be successful in lower secondary school as well. Through such an approach, all students might be able to learn insight into formulas and develop some confidence in their own reasoning with and about algebraic formulas.

REFERENCES

Anderson, J.R. (1980). *Cognitive psychology and its implications*. San Francisco, CA: W.H. Freeman and Company.

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics, 14*(3), 24–35.

Arcavi, A. (2005). Developing and using symbol sense in mathematics. *For the Learning of Mathematics*, *25*(2), 42–47.

Arcavi, A., Drijvers, P., & Stacey, K. (2017). *The teaching and learning of algebra: Ideas, insights and activities*. London / New York: Routledge.

Arievitch, I. M., & Haenen, J. P. (2005). Connecting sociocultural theory and educational practice: Galperin's approach. *Educational Psychologist*, *40*(3), 155–165.

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). The development of students' graphical understanding of the derivative. *The Journal of Mathematical Behavior*, *16*(4), 399–431.

Ayalon, M., Watson, A., & Lerman, S. (2015). Functions represented as linear sequential data: Relationships between presentation and student responses. *Educational Studies in Mathematics*, *90*(3), 321–339.

Arzarello, F., Bazzini, L., & Chiappini, G. (2001). A model for analysing algebraic processes of thinking. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (*Vol. 22,* pp. 61–81). Dordrecht, the Netherlands: Kluwer Academic Publishers.

Ball, L., Pierce, R., & Stacey, K. (2003). Recognising Equivalent Algebraic Expressions: An Important Component of Algebraic Expectation for Working with CAS. *International Group for the Psychology of Mathematics Education*, *4*, 15–22.

Barnard, A. D. & Tall, D. O. (1997). Cognitive units, connections and mathematical proof. In E. Pehkonen (Ed.), *Proceedings of the Twenty first International Conference for the Psychology of Mathematics Education (Vol.2,* pp. 41–48). Lahti, Finland.

Barsalou, L. W. (1992). Frames, concepts, and conceptual fields. In A. Lehrer & E. Kittay (Eds.), *Frames, fields, and contrasts: New essays in semantic and lexical organization.* (pp. 21–74). Hillsdale, NJ, England: Lawrence Erlbaum Associates, Inc.

Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology*, *59*, 617–645.

Bills, L, Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006, July). Exemplification in mathematics education. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the*

Psychology of Mathematics Education (*Vol. 1*, pp. 126–154). Prague, Czech Republic: Charles University.

Berliner, H., & Ebeling, C. (1989). Pattern knowledge and search: the SUPREM architecture. *Artificial Intelligence*, *38*(2), 161–198.

Bredeweg, B., & Forbus, K. D. (2003). Qualitative modeling in education. *AI magazine*, *24*(4), 35–35.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*(3), 247–285.

Burkhardt, H., & Swan, M. (2013). Task design for systemic improvement: principles and frameworks. In C. Margolinas (Ed.), *Task design in mathematics education*: *Proceedings of ICMI Study 22* (pp. 431–439). Oxford, United Kingdom: University of Oxford.

Campitelli, G., & Gobet, F. (2010). Herbert Simon's decision-making approach: Investigation of cognitive processes in experts. *Review of General Psychology*, *14*(4), 354–364.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *22*(5), 352–378.

Carlson, M. P., Madison, B., & West, R. D. (2015). A study of students' readiness to learn calculus. *International Journal of Research in Undergraduate Mathematics Education*, *1*(2), 209–233.

Chazan, D., & Yerushalmy, M. (2003). On appreciating the cognitive complexity of school algebra: Research on algebra learning and directions of curricular change. In J. Kilpatrick, W. G. Martin, & D. Shifter (Eds.), *A research companion to Principles and standards for school mathematics* (pp. 123–135). Reston, VA: NCTM.

Chi, M. T., Feltovich, P. J., & Glaser, R. (1981). Categorization and representation of physics problems by experts and novices. *Cognitive Science*, *5*(2), 121–152.

Chi, M. T., Glaser, R., & Rees, E. (1982). Expertise in problem solving. In R. Sternberg (Ed.), *Advances in the psychology of human intelligence* (pp.7–77). Hillsdale, NJ: Erlbaum.

Chi, M.T.H. (2006). Methods to assess the representations of experts' and novices' knowledge. In K. A. Ericsson, N. Charness, P. Feltovich, and R. R. Hoffman, R. R. (Eds.), *Cambridge handbook of expertise and expert performance* (pp. 223–242). Cambridge, United Kingdom: Cambridge University Press.

Chi, M. T. H. (2011). Theoretical perspectives, methodological approaches, and trends in the study of expertise. In Y. Li, & G. Kaiser (Eds.), *Expertise in mathematics instruction: Advancing* *research and practice from an international perspective* (pp. 17–39). New York, NY: Springer.

Collins, A. (2006). Cognitive apprenticeship. In R. K. Sawyer (Ed.), *The Cambridge Handbook of the Learning Sciences* (pp. 47–60). Cambridge, United Kingdom: Cambridge University Press.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, *26*(1), 66–86.

Crowley, L., & Tall, D. O. (1999). The roles of cognitive units, connections and procedures in achieving goals in college algebra. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of PME, (Vol. 2*, pp. 225–232). Rotterdam, the Netherlands: Sense.

Davis, R. B. (1983). Diagnosis and evaluation in mathematics instruction: Making contact with students' mental representations. In C. Smith (Ed.), *Essential knowledge for beginning educators* (pp. 101–111). Washington, DC: American Association of Colleges of Teacher Education.

De Corte, E. (2010). Historical developments in the understanding of learning. In D. Hanna, I. David, & B. Francisco (Eds.), *Educational research and innovation the nature of learning using research to inspire practice: Using research to inspire practice* (pp. 35–60). Paris, France: OECD Publishing.

De Groot, A. D. (1965). *Thought and choice in chess*. The Hague, The Netherlands: Mouton.

De Groot, A. D., Gobet, F., & Jongman, R. W. (1996). *Perception and memory in chess: Studies in the heuristics of the professional eye*. Assen, the Netherlands: Van Gorcum & Co.

De Jong, T., & Ferguson-Hessler, M. G. (1986). Cognitive structures of good and poor novice problem solvers in physics. *Journal of Educational Psychology*, *78*(4), 279–288.

Derry, S. J. (1996). Cognitive schema theory in the constructivist debate. *Educational Psychologist*, *31*(3-4), 163–174.

Drijvers, P. (1996). L'Hopital's weight problem and the TI-92. *The SAC Newsletter, 1*, 71–80.

Drijvers, P., & Doorman, M. (1996). The graphics calculator in mathematics education. *The Journal of Mathematical Behavior*, *15*(4), 425–440.

Drijvers, P. (2002). Learning mathematics in a computer algebra environment: obstacles are opportunities. *Zentralblatt für Didaktik der Mathematik*, *34*(5), 221–228.

Drijvers, P. (2003). *Learning algebra in a computer algebra environment: Design research on the understanding of the concept of parameter*. Utrecht, the Netherlands: CD Beta Press.

Drijvers, P., Goddijn, A., & Kindt, M. (2011). Algebra education: Exploring topics and themes. In P. Drijvers (Ed.), *Secondary algebra education. Revisiting topics and themes and exploring the unknown* (pp. 5–26). Rotterdam, Boston, Taipei: Sense Publishers.

Drijvers, P., Van Streun, A., & Zwaneveld, B. (Red.) (2012). *Handboek Wiskundedidactiek.* Utrecht, the Netherlands: Epsilon.

Drijvers., P., & Kop, P. (2012). Variabelen en vergelijkingen. In P. Drijvers, A. van Streun & B. Zwaneveld (Eds.), *Handboek wiskundedidactiek* (pp. 53–83). Amsterdam, the Netherlands: Epsilon.

Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 3–26). Cuernavaca, Morelos.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*(1–2), 103–131.

Eisenberg, T., & Dreyfus, T. (1994). On understanding how students learn to visualize function transformations. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education* (*Vol. 1*, pp. 45–68). Providence, RI: American Mathematical Society.

Ericsson, K. A. (2006). Protocol analysis and expert thought: Concurrent verbalizations of thinking during experts' performance on representative tasks. In Ericsson, K.A. Charness, N., Feltovich, P.J., & Hoffman, R.R. (Eds.), *The Cambridge handbook of expertise and expert performance* (pp. 223–241). Cambridge, United Kingdom: Cambridge University Press.

Ernest, P. (1990). The meaning of mathematical expressions: Does philosophy shed any light on psychology? *British Journal for Philosophy of Science 41,* 443–460.

Even, R. (1998). Factors involved in linking representations of functions. *The Journal of Mathematical Behavior*, *17*(1), 105–121.

Eysenck, M. W., & Keane, M. T. (2015). *Cognitive psychology: A student's handbook*. Taylor & Francis, Philadelphia, PA.

Fey, J. T. (1990). 'Quantity', in L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 61–94). Washington, D.C.: National Academy Press.

Field, A. P. (2012). *Discovering statistics using SPSS (4th ed.).* London: Sage Publications Ltd.

Forbus, K. D. (1996). Qualitative Reasoning. In A. B. Tucker (Ed.), *The Computer Science and Engineering Handbook* (pp. 715–733). Boca Raton, FL: CRC.

Friedlander, A., & Arcavi, A. (2012). How to practice it: An integrated approach to algebraic skills. *Mathematics Teacher*, *105*(8), 608–614.

Gobet, F. (1997). A pattern-recognition theory of search in expert problem solving. *Thinking & Reasoning*, *3*(4), 291–313.

Gobet, F., & Simon, H. A. (1996). The roles of recognition processes and look-ahead search in time-constrained expert problem solving: Evidence from grand-master-level chess. *Psychological science*, *7*(1), 52–55.

Gobet, F. (1998). Chess players' thinking revisited. *Swiss Journal of Psychology, 57,*18–32.

Goldenberg, E. P. (1988). Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions. *The Journal of Mathematical Behavior, 7*(2), 135–173.

Goldenberg, E., Lewis, P., O'Keefe, J. (1992). Dynamic representation and the development of a process understanding of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 235–260). Mathematical Association of America, Washington, D.C.

Goldenberg, P., & Mason, J. (2008). Shedding light on and with example spaces. *Educational Studies in Mathematics*, *69*(2), 183–194.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education*, *25*(2), 116–140.

Heid, M. K., Thomas, M. O. J., & Zbiek, R. M. (2013). How might computer algebra systems change the role of algebra in the school curriculum. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third International Handbook of Mathematics Education* (pp. 597–641). New York, NY: Springer.

Hennessy, S., Fung, P., & Scanlon, E. (2001). The role of the graphic calculator in mediating graphing activity. *International Journal of Mathematical Education in Science and Technology*, *32*(2), 267–290.

Hoch, M., & Dreyfus, T. (2005). Students' difficulties with applying a familiar formula in an unfamiliar context. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (*Vol. 3*, pp. 145–152). Melbourne: PME.

Hoch, M., & Dreyfus, T. (2010). Developing Katy's algebraic structure sense, In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the 6th Congress of the European Society for Research in Mathematics* (pp. 529–538), Lyon: CERME.

Janvier, C. (1987). *Problems of representation in the teaching and learning mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Jonassen, D. H., Beissner, K., & Yacci, M. (1993). *Structural knowledge: Techniques for representing, conveying, and acquiring structural knowledge*. Hillsdale: Lawrence Erlbaum Associates.

Kaput, J. J. (1998). Representations, inscriptions, descriptions and learning: A kaleidoscope of windows. *The Journal of Mathematical Behavior*, *17*(2), 265–281.

Keller, B. J. (1994). *Symbol sense and its development in two computer algebra system environments*. (Western Michigan University, 1993). Dissertation Abstracts International, *54*: 5704.

Kenney, R. (2008) *The influence of symbols on pre-calculus students' problem solving goals and activities*. Doctoral dissertation. North Carolina State University. https://pqdtopen.proquest.com/doc/304553264.html?FMT=ABS.

Kieran, C. (2006). Research on the learning and teaching of algebra. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and Future* (pp. 11–49). Rotterdam: Sense.

Kieran, C., & Drijvers, P. (2006). The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection: A study of CAS use in secondary school algebra. *International Journal of Computers for Mathematical Learning*, *11*(2), 205–263.

Kilpatrick, J., & Izsak, A. (2008). A history of algebra in the school curriculum. In C.E. Greens & R. Rubenstein (Eds.), *Algebra and algebraic Thinking in school mathematics, seventieth yearbook* (pp. 3–18). Reston, VA: NCTM.

Kilpatrick, J., Swafford, J. O., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, D.C.: National Academy Press.

Kindt, M. (2011). Principles of practice. In P. Drijvers (Ed.), *Secondary algebra education. Revisiting topics and themes and exploring the unknown* (pp. 137–178). Rotterdam, Boston, Taipei: Sense Publishers.

Kirschner, P.A., & Van Merriënboer, J.J.G. (2008). Ten steps to complex learning: A new approach to instruction and instructional design. In T. L. Good (Ed.), *21st Century Education: A Reference Handbook* (pp. 244–253). Thousand Oaks, CA: Sage.

Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education*, *31*(4), 500–507.

Kop, P., Janssen, F., Drijvers, P., Veenman, M., & Van Driel, J. (2015). Identifying a framework for graphing formulas from expert strategies. *The Journal of Mathematical Behavior*, *39*, 121–134.

Kop, P., Janssen, F., Drijvers, P., & van Driel, J. (2017). Graphing formulas: Unraveling experts' recognition processes. *The Journal of Mathematical Behavior*, *45*, 167–182.

Kop, P., Janssen, F., Drijvers, P., & van Driel, J. (2020a). Promoting insight into algebraic formulas through graphing by hand. *Mathematical Thinking and Learning*, 1-20. https://doi.org/10.1080/10986065.2020.1765078

Kop, P., Janssen, F., Drijvers, P., & van Driel, J. (2020b). The relation between graphing formulas by hand and students' symbol sense. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-020-09970-3

Landa, L.N. (1983). Descriptive and prescriptive theories of learning and instruction: an analysis of their relationships and interactions. In C.M. Reigeluth (Ed.), *Instructional-design theories and models: An overview of their current status* (pp. 55–73). Hillsdale, NJ: Lawrence Erlbaum Associates.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, *60*(1), 1–64.

Lesh, R. (1999). The development of representational abilities in middle school mathematics. In I. E. Sigel (Ed.), *Development of Mental Representation: Theories and Application* (pp. 323–350). Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers.

Marshall, S. P. (1995). *Schemas in problem solving*. Cambridge, United Kingdom: Cambridge University Press.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London, United Kingdom: Routledge.

Mason, J. (2003). On the structure of attention in the learning of mathematics. *Australian Mathematics Teacher, 59*(4), 17–25.

Mason, J. (2004/2008). Doing ≠ Construing and Doing + Discussing ≠ Learning: The Importance of the Structure of Attention. ICME 10 Regular Lecture. In M. Niss (Ed.), *Proceedings of ICME 10 CD*. Roskilde, Denmark: IMFUFA.

Merrill, M. D. (2013). *First principles of instruction: Identifying and designing effective, efficient and engaging instruction*. Hoboken, NJ: Pfeiffer/John Wiley & Sons.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 782–789). Pittsburgh, PA: RUME.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 69–100). Hillsdale, NJ: Lawrence Erlbaum Associates.

NCTM–National Council of Teachers in Mathematics (2000). *Principles and standards for school mathematics*. http://www.nctm.org/standards.

Nistal, A. A., Van Dooren, W., Clarebout, G., Elen, J., & Verschaffel, L. (2009). Conceptualising, investigating and stimulating representational flexibility in mathematical problem solving and learning: a critical review. *ZDM—The International Journal on Mathematics Education*, *41*(5), 627–636.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27–42). Washington, D.C.: Mathematical Association of America.

Philipp, R., & Martin, W. & Richgels, G. (1993). Curricular implications of graphical representations of functions. In T.A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 239–278). Hillsdale, NJ: Lawrence Erlbaum Associates.

Pierce, R., & Stacey, K. (2004). Monitoring Progress in Algebra in a CAS Active Context: Symbol Sense, Algebraic Insight and Algebraic Expectation. *International Journal for Technology in Mathematics Education*, *11*(1), 3–11.

Pierce, R., & Stacey, K. (2007). Developing algebraic nsight. *Mathematics Teaching Incorporating Micromath*, *203*, 12–16.

Polya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.

Presmeg, N. C. (1998). Metaphoric and metonymic signification in mathematics. *The Journal of Mathematical Behavior*, *17*(1), 25–32.

Radford, L. (2004). Syntax and meaning. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (*Vol. 1*, pp. 161–166). Bergen, Norway: PME.

Ruiz-Primo, M. A., & Shavelson, R. J. (1996). Problems and issues in the use of concept maps in science assessment. *Journal of Research in Science Teaching*, *33*(6), 569–600.

Ruthven, K. (1990). The influence of graphic calculator use on translation from graphic to symbolic forms. *Educational Studies in Mathematics, 21*(5), 431–450.

Ruthven, K., Deaney, R., & Hennessy, S. (2009). Using graphing software to teach about algebraic forms: A study of technology-supported practice in secondary-school mathematics. *Educational Studies in Mathematics*, *71*(3), 279–297.

Saariluoma, P. (1992). Error in chess: The apperception-restructuring view. *Psychological Research*, *54*(1), 17–26.

Sandefur, J., Mason, J., Stylianides, G. J., & Watson, A. (2013). Generating and using examples in the proving process. *Educational Studies in Mathematics*, *83*(3), 323–340.

Schoenfeld, A. H. (1978). Presenting a strategy for indefinite integration. *The American Mathematical Monthly*, *85*(8), 673–678.

Schoenfeld, A.H. (1985). Cognitive Science and Mathematics Education. In A. Schoenfeld (Ed.), *Cognitive Science and Mathematical Education*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York, NY: Mcmillan.

Schwarz, B. B., & Hershkowitz, R. (1999). Prototypes: Brakes or levers in learning the function concept? The role of computer tools. *Journal for Research in Mathematics Education, 30*(4), 362–389.

Schwartz, J., & Yerushalmy, M. (1992). Getting students to function on and with algebra. In E. Dubinsky, & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 261–289). Washington, DC: Mathematical Association of America. https://www.researchgate.net/publication/266439744.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36.

Sfard, A. & Linchevski, L. (1994). The gains and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics, 26*, 191–228.

Slavit, D. (1997). An alternate route to the reification of function. *Educational Studies in Mathematics*, *33*(3), 259–281.

Star, J. R., & Stylianides, G.J. (2013). Procedural and Conceptual Knowledge: Exploring the Gap Between Knowledge Type and Knowledge Quality. *Canadian Journal of Science, Mathematics, and Technology Education, 13*(2), 169–181.

Stylianou, D.A. (2002). On the interaction of visualisation and analysis: the negotation of a visual representation in expert problem solving. *Journal of Mathematical Behavior, 21*, 303–317.

Stylianou, D. A., & Silver, E. A. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning*, *6*(4), 353–387.

Stylianou, D. A. (2011). An examination of middle school students' representation practices in mathematical problem solving through the lens of expert work: Towards an organizing scheme. *Educational Studies in Mathematics*, *76*(3), 265–280.

Swan, M. (2005). *Standards Unit-Improving learning in mathematics: challenges and strategies*. Nottingham, United Kingdom: University of Nottingham. https://www.ncetm.org.uk/files/224/improving_learning_in_mathematicsi.pdf.

Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design. *Learning and Instruction*, *4*(4), 295–312.

Sweller, J., van Merriënboer, J. J., & Paas, F. (2019). Cognitive architecture and instructional design: 20 years later. *Educational Psychology Review, 31*(2), 1–32.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.

Thomas, M. O., Wilson, A. J., Corballis, M. C., Lim, V. K., & Yoon, C. (2010). Evidence from cognitive neuroscience for the role of graphical and algebraic representations in understanding function. *ZDM—The International Journal on Mathematics Education*, *42*(6), 607–619.

Thomas, M. O. J., & Hong, Y. Y. (2001). Representations as conceptual tools: Process and structural perspectives, In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of The 25th Conference of the International Group for the Psychology of Mathematics Education* (pp. 257–264). Utrecht, the Netherlands.

Thompson, P. W. (2013). In the absence of meaning…. In Leatham, K. (Ed.), *Vital directions for mathematics education research* (pp. 57–93). New York, NY: Springer.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: NCTM.

Tsamir, P., & Bazzini, L. (2004). Consistencies and inconsistencies in students' solutions to algebraic 'single-value' inequalities. *International Journal of Mathematical Education in Science and Technology*, *35*(6), 793–812.

Turşucu, S., Spandaw, J., & de Vries, M. J. (2018). Search for Symbol Sense Behavior: Students in Upper Secondary Education Solving Algebraic Physics Problems. *Research in Science Education, 48*(5), 1–27.

Van Merriënboer, J. J., Clark, R. E., & De Croock, M. B. (2002). Blueprints for complex learning: The 4C/ID-model. *Educational technology research and development*, *50*(2), 39–61.

Van Stiphout, I. M., Drijvers, P., & Gravemeijer, K. (2013). The development of students' algebraic proficiency. *International Electronic Journal of Mathematics Education*, *8*(2–3), 62–80.

Van Streun, A. (1989). *Heuristisch wiskunde onderwijs*. Dissertation. Groningen, the Netherlands: University Press.

Veenman, M. V., Van Hout-Wolters, B. H., & Afflerbach, P. (2006). Metacognition and learning: Conceptual and methodological considerations. *Metacognition and Learning*, *1*(1), 3–14.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(5), 356–366.

Watson, A., & Mason, J. (2002). Extending example spaces as a learning/teaching strategy in mathematics. In A. Cockburn & E. Nardi (Eds.). *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (*Vol. 4*, pp. 377–385). Norwich, United Kingdom: PME.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Lawrence Erlbaum, Inc.

Wijers, M & Kemme, S. (2000). Welke algebra heb je nodig in 4 VWO? *Nieuwe Wiskrant, 20*(1), 22–25.

Yerushalmy, M. (1997). Reaching the unreachable: Technology and the semantics of asymptotes. *International Journal of Computers for Mathematical Learning*, *2*(1), 1–25.

Yerushalmy, M., & Gafni, R. (1992). Syntactic manipulations and semantic interpretations in algebra: The effect of graphic representation. *Learning and Instruction*, *2*(4), 303–319.

Zandieh, M.J., & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning: The concept of derivative as an example. *The Journal of Mathematical Behavior, 25*(1), 1–17.

Zbiek, R. M., & Heid, M. K. (2011). Using technology to make sense of symbols and graphs and to reason about general cases. In T. Dick & K. Hollebrands (Eds.), *Focus on reasoning and sense making: Technology to support reasoning and sense making* (pp. 19–31). Reston, VA: NCTM.

SUMMARY

Students in, and even beyond, secondary school continue to have serious problems with algebra, in particular in giving meaning to algebraic formulas which are very abstract for them (e.g. Kieran, 2006). Many students lack symbol sense, that is, they have trouble with reading through formulas, recognizing the structure of formulas, and making sense of formulas. In many curricula, the importance of symbol sense is acknowledged (e.g., NCTM, 2000). The main aim of the present research was to promote aspects of students' symbol sense that enable students in grade 11 and 12 to read through formulas and to make sense of these formulas, and to deal with non-routine algebraic problems.

In chapter 1, we elaborate on the concept symbol sense, describe our strategy to teach this symbol sense to students in upper secondary school and give an outline of the different studies. Symbol sense is a very broad concept, which was described by Arcavi as "an intuitive feel for when to call on symbols in the process of solving a problem, and conversely, when to abandon a symbolic treatment for better tools" (Arcavi, 1994, p. 25). Drijvers et al. (2011) see symbol sense as complementary to basic skills, like procedural work, with a local focus and algebraic calculations. Symbol sense forms a compass for basic skills and is about taking a global view, algebraic reasoning, and adopting a strategic approach. Pierce and Stacey (2004) use the concept 'algebraic insight' for interpreting and making sense of algebraic calculations that are performed via computer algebra systems and therefore include manipulations of formulas to determine equivalence of formulas. As our focus was exclusively on reading through and making sense of algebraic formulas and not on manipulating them, we use the term insight into algebraic formulas, defined as the ability to recognize the structure of a formula and its components, and to reason with and about formulas.

To give meaning to algebraic formulas, Kieran (2006) and Radford (2004) have suggested to use linking multiple representations, like table, graph, formula, and realistic context. However, except for linear and exponential formulas, linking formulas to realistic contexts is in general difficult. For our research, we chose to link formulas to graphs. Although it is recommended to use graphing tools such as graphic calculators for learning about functions and their representations (Hennessy et al., 2001; Heid et al., 2013), Goldenberg (1988) suggested to use graphing by hand to establish a better connection between formulas and graphs. The need for pen-and-paper activities was later found by others (Kieran & Drijvers, 2006; Arcavi et al., 2017). As students in upper secondary school have experience with graphing tools, we followed Goldenberg's suggestion and focused on graphing formulas by hand, without technology (graphing formulas). When graphing formulas, the formulas are linked to its graphs. Graphs give a Gestalt-view of a function, visualizing the "story" a function tells in a single picture, and so emphasize the function object character and show how the dependent and independent variables covary in relation to each other. In this way, several aspects that seem problematic in learning about functions and formulas are addressed: mathematical objects like functions are not directly accessible as physical objects, switching between the process—object character of a function (seeing a function both as an input-output machine and as an object (Moschkovich et al., 1993)), and covariational reasoning (coordinating how two varying quantities change in relation to each other (Carlson et al., 2002)).

To engage in algebra, a combination of basic skills and symbol sense is needed. However, it is hard to teach symbol sense (Arcavi et al., 2017; Hoch & Dreyfus, 2005). In this dissertation, we tried to promote students' insight into formulas and chose to teach graphing formulas for this purpose. The overall research question of this thesis is:

How can teaching graphing formulas foster grade 11 and 12 students' insight into formulas and their symbol sense to solve non-routine algebraic problems?

We conducted four studies to investigate this overall research question. Because it was not clear what knowledge and skills are needed to graph formulas effectively and efficiently, in studies 1 and 2 (chapter 2 and 3) we first investigated expert behavior and thinking in graphing formulas. The findings resulted in a framework that guided the intervention in study 3. In study 3 (chapter 4), we designed an intervention to teach grade 11 students' expertise in graphing formulas, that is, graphing through a combination of recognition and qualitative reasoning and investigated whether students' insight into algebraic formulas was promoted. In study 4 (chapter 5), we focused on the relation between students' symbol sense involved in graphing formulas and in solving algebraic problems.

In chapter 2, we investigate experts' strategies in graphing formulas. Expertise literature indicates that problem solving can be described in terms of recognition and heuristic search (Chi, 2011; Gobet, 1998; Gobet & Simon, 1996). To describe experts' strategies in graphing formulas, a two-dimensional framework was proposed, using levels of recognition and heuristics. The levels of recognition reflect the levels of awareness formulated by Mason (2003): from complete recognition and instantly knowing the graph, to decomposing the formula into manageable sub-formulas, to perceiving graph properties, to no recognition at all and only calculating some points. On each level of recognition, domain-specific heuristics were described and ordered from strong to weak. Strong heuristics give information about large parts of a graph, like using qualitative reasoning about the function's infinity behavior or when adding or multiplying two sub-graphs. Weak heuristics only give local information about the graph, like calculating a point of a graph. Two research questions guided this study: Does the framework describe strategies in graphing formulas appropriately and discriminatively? Which strategies do experts use in tasks graphing formulas?

In this case study, five experts in mathematics and three secondary-school math teachers thought aloud while graphing a more complex function ($y = 2x\sqrt{8-x} - 2x$) and had to find a formula that would fit a given graph. The video recordings were transcribed, cut into fragments which contained crucial steps of explanations, and analyzed. The results showed that all these steps could be encoded within the two-dimensional framework, generating paths in the framework. We concluded that the framework was discriminative, because different strategies by the participants gave different paths in the framework. The experts used various strategies when graphing formulas: some focused on their repertoire of formulas they could instantly visualize by a graph; others relied on strong heuristics, such as qualitative reasoning. The experts' main strategies were: recognizing function families and using their prototypical graphs, recognizing key graph features, using qualitative reasoning when exploring parts of the graph, e.g., infinity behavior or when composing two sub-graphs after decomposing a formula into two sub-formulas. The teachers hardly used function families and more often used weaker heuristics. It was concluded that expertise in graphing formulas does not involve calculations of derivatives, as all our experts seemed to hesitate to start such calculations and made mistakes when they did.

In chapter 3, we report on the study in which we investigated experts' recognition processes in graphing formulas. We focused on instantly graphable formulas. An instantly graphable formula (IGF) is a formula that a person can instantly visualize by a graph. IGFs can be seen as building blocks in thinking and reasoning with and about formulas and graphs. These building blocks can be combined (addition, multiplication, chaining, etc.) into new and more complex building blocks (e.g., IGFs $y = -x^4$ and $y = 6x^2$ can be combined into a 4-degree polynomial function $y = -x^4 + 6x^2$). The research questions in this study were: Can we describe experts' repertoires of instant graphable formulas (IGFs) using categories of function families? What do experts attend to when linking formulas and graphs of IGFs, described in terms of prototype, attribute, and part-whole reasoning? The five experts of study 1 worked on a card-sorting task to investigate what function families experts use, a matching task to investigate experts' recognition, and a thinking aloud multiple-choice task to portray experts' recognition processes. The experts' results in the card-sorting task showed that the categories they constructed, and the category descriptions, were very similar, although some experts made more sub-categories (e.g., differences between parabolas with a max versus with a min). These descriptions were closely related to the basic function families that are taught in secondary school: linear functions, polynomial functions, exponential and logarithmic functions, broken functions, and power functions. The experts had no problems with the matching task, in which they had to match formulas with one of the 21 alternative graphs.

The analyses of the thinking aloud protocols of the multiple-choice tasks were based on both Barsalou's model of organized hierarchical knowledge (Barsalou, 1992) and on Schwarz and Hershkowitz's (1999) descriptions of concept images, using prototypicality (the use of prototypical members of a category or function family), attribute understanding (the ability to recognize attributes of a function across representations), and part–whole reasoning (the ability to recognize that different formulas and/or different graphs relate to the same entity). The findings of these analyses suggested that experts' recognition of IGFs can be described with the Barselou model in which formulas, function families, prototypes, a set of attributes and values, and graphs are linked in well-connected hierarchical mental networks.

In chapter 4, we investigate how graphing formulas based on recognition and reasoning could be taught to grade 11 students with the aim to promote students' insight into algebraic formulas. The research question addressed was: How can grade 11 students' insight into algebraic formulas be promoted through graphing formulas? In an intervention of five 90-minute lessons, 21 grade 11 students were taught to graph formulas by hand. The intervention's design was based on experts' strategies in graphing formulas, that is, using a combination of recognition and qualitative reasoning. We used the principles of teaching complex skills, that is, using a whole task approach, with support and reflection tasks (Kirschner & Van Merriënboer, 2008; Merrill, 2013; Van Merriënboer et al., 2002), and also included the meta-heuristic "questioning the formula" (Landa, 1983; Pierce & Stacey, 2007). The five whole tasks reflected the levels of recognition of the two-dimensional framework. First, attention was paid to a repertoire of basic function families with their characteristics. Then, single transformations of prototypes of the function families were addressed. In the third whole task, students practiced decomposing a formula into two sub-formulas and composing the sub-graphs. In the fourth whole task, the focus was on the recognition of graph features from a formula, e.g., the zeroes and extreme values. In the last whole task, students explicitly practiced qualitative reasoning about infinity behavior, weaker and stronger components of a formula, in- and decreasing of functions, etc. Qualitative reasoning is often used by experts and is characterized by its focus on the global shape of the graph, with global descriptions and ignoring what is not relevant.

The students did a written pre- and post-test, followed by a retention test after four months, which contained a graphing task and a matching task that was similar to the one used in chapter 3. Six students were asked to think aloud during the graphing tasks in the pre- and post-test. The pre-test results showed that the students lacked insight into formulas, and the thinking-aloud protocols suggested a lack of recognition and reasoning skills. The post-test results showed that students had improved their recognition of function families and graph features as well as their qualitative reasoning abilities. In a post-intervention questionnaire, the students themselves indicated that they understood formulas better. In the retention test, the scores on the graphing task and multiple-choice task were, as expected, lower than in the post-test, but significantly higher than in the pre-test. This suggested a long-lasting effect of the intervention. The findings of this study suggested that, although many students still had problems with more complex formulas, teaching graphing formulas to grade 11 students, based on recognition and qualitative reasoning, might be a means to promote student insight into algebraic formulas in a systematical way.

In chapter 5, we explore the relation between students' graphing abilities and their symbol sense abilities to solve non-routine algebraic tasks, like: How many solutions does this equation have? What y-values can this formula have? To solve these kinds of problems, students could use their graphing abilities, but also other aspects of symbol sense, like abandoning the symbolic representation, and using graphs and/or reasoning, instead of starting calculations. So, the symbol sense involved in graphing formulas is a subset of the symbol sense involved in solving these algebra tasks. We investigated whether students might be able to use the symbol sense involved in graphing formulas in other non-routine algebraic problems that could be solved with graphs and reasoning. The main research question of this study was: How do grade 12 students' abilities to graph formulas by hand relate to their use of symbol sense while solving non-routine algebra tasks? Two sub-questions were formulated: To what extent are students' graphing formulas by hand abilities positively correlated to their abilities to solve algebraic tasks with symbol sense? Is students' use of symbol sense in graphing formulas similar or different from their use of symbol sense in solving non-routine algebraic tasks? A written symbol sense test was administered to a group of 114 grade 12 students, including 21 students who had participated in the intervention described in chapter 4, and 93 students from five other schools across the Netherlands. Six students who were involved in the intervention were asked to think aloud during the symbol sense test, which consisted of 8 graphing tasks and 12 non-routine algebraic tasks. The results of the written test were graded, and the symbol sense use was analyzed and graded using four categories: blank, calculations, making a graph, recognition, and reasoning.

A positive correlation was found between students' graphing abilities and their abilities to solve algebra tasks and their symbol use when solving these tasks, also when corrected for students' general math abilities. Students who scored high on the graphing tasks did more often use the strategy "making a graph" when working on the algebra tasks. With respect to the second sub-question, we found that 16 of the 21 students involved in the teaching of graphing formulas by hand in the intervention of study 3 belonged to the 25% highest scoring students on the graphing tasks. These high scoring students used more symbol sense when solving non-routine algebra tasks than the other students. The six thinking-aloud students who were among these 16 students showed that they used similar aspects of symbol sense in both the graphing tasks and the algebra tasks, including combinations of recognition of function families and key graph features and qualitative reasoning. These findings seemed to confirm our expectations that students who are able to graph formulas by hand can use these abilities in a broader domain of non-routine algebra tasks.

In the concluding chapter 6, we first present the findings of the four separate studies, followed by discussion and conclusions with the main contributions and limitations of the studies. The main aim of this research was to promote aspects of students' symbol sense, that is, students' abilities to read through formulas, to make sense of formulas, and to use this symbol sense when solving algebraic problems. The premise in our study was that students have to make sense of algebraic formulas and, therefore, have to be able to read through them. If students cannot make sense of their algebra activities, they will not develop confidence in their algebraic work, which result in a reluctancy to engage in algebraic reasoning and thinking. To enable students to develop expertise and confidence in reading algebraic formulas, we selected a small but rich domain in algebra, namely graphing formulas. Graphing formulas requires students to read through many kinds of formulas, and it allows them to make sense of these formulas by linking formulas to their graphs. As our aim was to foster insight into formulas, we restricted the tasks to interpreting formulas and ignored algebraic manipulations, which are often at the core of regular algebra education and a source of problems for many students. We chose to graph formulas by hand because connections between formula and graph established via by hand activities are more effective than via computer graphing (Goldenberg, 1988). As experts are supposed to use insight into formulas, we investigated expert behavior in graphing formulas, identified essential thinking processes, and described these in terms of recognition and reasoning in a two-dimensional framework. This gave us a clue about what to teach. Based on the two-dimensional framework, we designed an intervention of five lessons of 90 minutes, the so-called GQR-design (Graphing based on Qualitative reasoning and Recognition). Through whole tasks, with help and reflection questions, and using "questioning the formula" as a leading meta-heuristic, graphing formulas was taught step-by-step and in a systematical way. In this GQR-design, explicit attention is paid to the interplay between recognition and reasoning by using combinations of function families with their prototypical graphs as building blocks, key graph features, and qualitative reasoning. The whole task approach forces students to take a global view for recognition, to reason and argue, and to consider their strategies, which are essential aspects of symbol sense (Drijvers et al., 2011). We expected that students could use these aspects of symbol sense once they had learned these through graphing formulas while solving non-routine algebra tasks. We designed a symbol sense test with non-routine algebra tasks that could be solved via recognition, reasoning, and making a graph. Results with this symbol sense test suggested that the students involved in the intervention were able to use their symbol sense in graphing formulas and were able to use graphs as visualizations while solving the non-routine algebra tasks. We concluded that teaching graphing formulas by hand with our GQR-design could be an effective means to teach students in the higher grades of secondary school aspects of symbol sense, like insight into algebraic formulas, that can be used to solve non-routine algebra tasks.

Inevitably, the studies reported have their limitations. In chapter 2 and 3, only five experts participated and worked on only two tasks, due to the labor-intensive method for strategy assessment. Although we expected that most common strategies were captured in the two-dimensional framework, testing a larger group of potential experts before describing expertise in graphing formulas might give another, more detailed picture of expertise. In the intervention in study 3, only one class of students from the Netherlands was involved, and no comparison group was included. However, one year and two years later, the same series of lessons from the intervention was used in two other groups in the same school, both of 23 students. Both groups made the same post-test that was used in our study 3, and the scores showed similar results. Although this might be a confirmation that students can develop insight into formulas via our GQR-design, we suggest future research including more students and teachers to further investigate whether and how students can improve their insight into formulas through GQR-design.

In the symbol sense test in study 4, we used a combination of graphing tasks and algebra tasks for research purposes. This might have suggested to use "making a graph" when working on the algebra tasks. In the test, we often used the variables x and y to make the test more recognizable for the students. In future studies, we suggest using other variables than x and y more often. The algebra tasks in the current test were limited to those that could be solved using graphing and reasoning. In future tests, we suggest broadening this scope. Algebraic problems often require a combination of reasoning, graphing and calculations, and a next step might be to also include tasks in which one has to consider whether calculations are required or not, like "how many solutions does the equation $3.6(1 - \mathrm{e}^{-2.5t}) = 10t$ have?" In the symbol sense test, we found that many students had problems with solving inequalities such as $x(x - 1) > 4x$. Instead of using their graphing abilities, half of the best graphing students started calculations and were often unsuccessful. These findings seemed to suggest that an inequality triggered previously learned associations, and that such associations might hinder later learned symbol sense. Further research is needed to investigate how just learned symbol sense can be incorporated in students' strategies and habits to deal with algebraic problems. In study 4, we suggested that graphing formulas based on recognition and reasoning might be a means to teach symbol sense in upper secondary school that could be used by students to solve non-routine algebra tasks. More research is needed to clarify this suggestion. A next step might be to set up a quasi-experimental study, in which a group of students is taught to graph formulas like in the intervention, using a control group and a pre-test and post-test. As we expect that difficulties with insight into algebraic formulas and symbol sense are not exclusive to Dutch students, students and teachers from other countries should also be included in future studies.

This dissertation contributes to our knowledge about symbol sense and teaching symbol sense. Firstly, it describes the nature of expertise in terms of recognition, reasoning, and its interplay, and shows how this can be elaborated for the domain of graphing formulas. Secondly, it shows how grade 11 students can acquire insight into algebraic formulas through an innovative intervention about graphing formulas. And, thirdly, it explores how symbol sense might be taught to students.

As a first contribution of this research, we described the nature of expertise in graphing formulas in terms of recognition and heuristics in a two-dimensional framework. We identified several levels of recognition which determines the problem space and, therefore, the heuristic search: recognition guides heuristic search. On each level of recognition, we formulated heuristics in the two-dimensional framework stressing the interplay between recognition and domain-specific heuristic search. This approach differs from, for instance, descriptions of knowledge bases, in which mathematical competences are described in lists of several components, like conceptual knowledge, procedural knowledge, and strategic competence. Although the need for integration of different components has been stressed (Kilpatrick, Swafford, & Findell, 2001), this has, to our knowledge, not led to models in which these components are actually integrated. Describing expertise in terms of recognition and heuristics in a two-dimensional framework also seems possible in other domains of algebra. Pouwelse, Janssen and Kop (submitted) proposed a framework with recognition and heuristics for finding indefinite integrals in calculus. The framework could be used as an instrument for designing teaching material but also as an instrument in teacher professional development. Further research is suggested to explore how the interplay between recognition and heuristic search in other domains like solving equations could be described and used in designing teaching and/or in teacher professional development.

As a second contribution of the research, we showed how grade 11 students can acquire insight into algebraic formulas through graphing formulas via our GQR-design. The GQR-design differs from both regular and other innovative approaches to learning about functions, in particular regarding the link between formulas and graphs. Our approach differs from regular education about functions, which often focuses on the manipulation of algebraic expressions (Arcavi et al., 2017; Schwartz & Yerushalmy, 1992) and on using graphing tools to explore function families and to work on calculus problems. In comparison to regular approaches, in the GQR-design, explicit attention is paid to recognition and to reasoning with and about functions. In our design, we use basic function families as building blocks for formulas, following Davis' suggestion to use larger thinking units to allow for better recognition of the formula's structure (Davis, 1983), and we pay attention to read key features from the formulas. Explicit attention is paid to reasoning, e.g., about parameters of function families, about infinity behavior, and when composing two sub-graphs. Other innovative approaches often focus on reasoning about functions, using graphing tools. In comparison to our approach, these approaches do not explicitly pay attention to qualitative reasoning and to the recognition and use of function families. The importance of qualitative reasoning and its omission in a mathematics curriculum was already stressed by Leinhardt et al. (1990), Goldenberg et al. (1992), Yerushalmy (1997), and Duval (2006), but to our knowledge, this qualitative reasoning has never been implemented in concrete and systematic teaching approaches.

The third contribution of this research is that it shows how symbol sense to solve non-routine algebra tasks might be taught to students. Symbol sense seems difficult to teach (Arcavi et al., 2017; Hoch & Dreyfus, 2005). In regular education, many teachers and students focus on basic skills and manipulating formulas and expressions (Arcavi et al., 2017), expecting that students will develop symbol sense through these kinds of practices. Innovative approaches focus more on reasoning, and give suggestions how to teach this, for instance through using productive practices, such as reverse thinking and constructing examples (Friedlander & Arcavi, 2012; Kindt, 2011), using rich, collaborative tasks (Swan, 2008), and snapshots for classroom discussions (Pierce & Stacey, 2007). At the core of our strategy is the idea of teaching symbol sense in a small domain of algebra, graphing formulas by hand, allowing students to develop expertise in this domain. If the teaching of graphing formulas focuses on essential aspects of symbol sense, like taking a global view for recognition, (qualitative) reasoning, and strategic work, then these essential aspects of symbol sense might be transferred to a broader domain of algebra. In our GQR-design, these essential aspects of symbol sense were explicitly and systematically taught as thinking tools, whereas in other approaches these thinking tools are often implicit. Our research shows that students obtained insight into formulas, and learned essential aspects of symbol sense, which they could later use while solving non-routine algebraic problems in the symbol sense test. The students involved in the intervention indicated that they thought they understood functions better, could visualize formulas better, in particular basic functions, and indicated that qualitative reasoning was very new and motivating for them (“we now use global reasoning; it is fun, this kind of reasoning”). This suggests that the GQR-design is a motivating and systematical way to teach students aspects of symbol sense.

The ability to read through formulas and make sense of them is an important aspect of symbol sense and will also remain important in the future when technology will take over manipulation of algebraic formulas even further. People will have to interpret results, make global estimations about results, and understand what is going on. For this purpose, they will need to develop some, what we might call, formula sense, that includes: making sense of a formula, using function families as building blocks of formulas, identifying and using the structure of a formula, and using qualitative reasoning. These ideas might be relevant for new curricula for secondary school. In this research, we showed how this symbol sense can be taught to students via our GQR-design.

SAMENVATTING

Formules schetsen om
symbol sense te bevorderen.
Vriendschap sluiten met
algebraïsche formules.

Leerlingen in het voortgezet onderwijs en zelfs daarna hebben nog steeds ernstige problemen met algebra, met name om betekenis te geven aan algebraïsche formules die voor hen zeer abstract zijn (bijv. Kieran, 2006). Veel leerlingen hebben weinig symbol sense, dat wil zeggen, dat ze moeite hebben met het doorzien van formules, het herkennen van de structuur van formules en het betekenis geven aan deze formules. In veel curricula wordt het belang hiervan erkend (bijv. NCTM, 2000). Het belangrijkste doel van ons onderzoek was het bevorderen van deze aspecten van de symbol sense die het studenten in staat stellen algebraïsche formules te doorzien en niet-standaard algebra problemen aan te pakken.

In het eerste hoofdstuk gaan we dieper in op het begrip symbol sense, beschrijven we onze strategie om deze symbol sense te onderwijzen aan leerlingen in de bovenbouw van het voortgezet onderwijs en geven we een overzicht van de verschillende studies. Symbol sense is een zeer breed begrip, dat door Arcavi werd beschreven als "een intuïtief gevoel voor wanneer en hoe wiskundige symbolen te gebruiken in het proces van het oplossen van een wiskundig probleem, en omgekeerd, wanneer een symbolische aanpak te stoppen en over te gaan op een andere aanpak" (Arcavi, 1994, p. 25). Drijvers et al. (2011) zien symbol sense als complementair aan basisvaardigheden, als procedureel werken en algebraïsch manipuleren. Zij zien symbol sense als een kompas voor de basisvaardigheden en daarbij gaat het om globaal kijken, algebraïsch redeneren, en strategisch werken. Pierce en Stacey (2004) gebruiken het begrip ‘algebraic insight’ voor het interpreteren en betekenis geven aan algebraïsche berekeningen die via computer algebra systemen zijn uitgevoerd. Hierbij speelt het bepalen van gelijkwaardigheid van formules en dus ook het manipuleren van formules een belangrijke rol. Aangezien onze focus lag op het doorzien van en betekenis geven aan algebraïsche formules en niet op het manipuleren van formules, gebruikten we de term inzicht in algebraïsche formules, gedefinieerd als het vermogen om de structuur van een formule en zijn componenten te herkennen, en te redeneren met en over formules.

Om betekenis te geven aan algebraïsche formules hebben Kieran (2006) en Radford (2004) gesuggereerd om gebruik te maken van meervoudige representaties van functies, zoals tabel, grafiek, formule en realistische context. Met uitzondering van lineaire en exponentiële formules is het koppelen van formules aan realistische contexten over het algemeen echter moeilijk. Voor ons onderzoek hebben we ervoor gekozen om formules te koppelen aan grafieken. In de literatuur wordt geadviseerd om gebruik te maken van technologie bij het leren over wiskundige functies waardoor formules eenvoudig omgezet kunnen worden in grafieken (Hennessy et al., 2001; Heid et al., 2013). Echter, Goldenberg (1988) suggereerde dat studenten het verband tussen formule en grafiek beter legden als grafieken met de hand getekend werden. De noodzaak van pen-en-papier activiteiten, naast het gebruik van technologie, werd later door anderen onderschreven (Kieran & Drijvers, 2006; Arcavi et al., 2017). Omdat leerlingen in de bovenbouw van het voortgezet onderwijs ervaring hebben met de grafische rekenmachine, hebben we de suggestie van Goldenberg gevolgd en gekozen om grafieken te laten schetsen met de hand, dus zonder technologie (schetsen van formules). Bij het schetsen van formules worden formules gekoppeld aan grafieken, die een Gestalt-view van een functie geven en het ‘verhaal’ dat een functie vertelt in een enkel beeld visualiseren. Grafieken benadrukken het object-karakter van een functie en laat de co-variatie van een functie zien, dat wil zeggen hoe de afhankelijke en onafhankelijke variabelen veranderen ten opzichte van elkaar. Op deze manier komen verschillende aspecten die problematisch lijken bij het leren over functies en formules aan bod: wiskundige objecten zoals functies zijn niet direct toegankelijk als fysieke objecten, het schakelen tussen het proces en object karakter van een functie, dat is een functie zien als een input-output machine en als een object (Moschkovich et al., 1993), en redeneren over de co-variatie van een functie (Carlson et al., 2002).

Om bekwaam te worden in algebra is een combinatie van basisvaardigheden en symbol sense nodig. Het is echter niet simpel om deze symbol sense te onderwijzen (Arcavi et al., 2017; Hoch & Dreyfus, 2005). In dit proefschrift hebben we geprobeerd het inzicht in formules bij leerlingen te bevorderen en hebben we ervoor gekozen om hiervoor het schetsen van formules te gebruiken. De algemene onderzoeksvraag in dit proefschrift was:

Hoe kan onderwijs in het schetsen van formules het inzicht van leerlingen in formules en hun symbol sense om niet-standaard algebraïsche problemen op te lossen bevorderen?

We hebben vier studies uitgevoerd om deze algemene onderzoeksvraag te kunnen beantwoorden. Omdat het niet duidelijk was welke kennis en vaardigheden nodig zijn om formules effectief en efficiënt te kunnen schetsen, hebben we in studie 1 en 2 (hoofdstukken 2 en 3) eerst het gedrag en denken van experts in bij het schetsen van grafieken onderzocht. De bevindingen resulteerden in een framework dat voor de interventie in studie 3 gebruikt werd. In studie 3 (hoofdstuk 4) ontwierpen we een interventie om leerlingen in VWO 5 het schetsen van formules door middel van een combinatie van herkenning en kwalitatief redeneren te onderwijzen, en we onderzochten of het inzicht van leerlingen in algebraïsche formules verbeterde. In studie 4 (hoofdstuk 5) richtten we ons op de relatie tussen de symbol sense van leerlingen bij het schetsen van formules en bij het oplossen van niet-standaard algebra problemen.

In hoofdstuk 2 beschrijven we het onderzoek naar de strategieën van experts bij het schetsen van formules. Uit de literatuur blijkt dat het oplossen van problemen beschreven kan worden in termen van herkenning en heuristisch zoeken (Chi, 2011; Gobet, 1998; Gobet & Simon, 1996). Om de strategieën van experts bij het schetsen van formules te beschrijven is een tweedimensionaal framework gepresenteerd, met niveaus van herkenning en heuristieken. De niveaus van herkenning kunnen gelinkt worden aan Mason's niveaus van 'awareness' (Mason, 2003) en variëren van volledige herkenning en het direct kennen van de grafiek, tot het ontleden van de formule in hanteerbare sub formules, tot het herkennen van enkele grafische eigenschappen, tot het ontbreken van enige herkenning. Op verschillende niveaus van herkenning worden in het framework domein specifieke heuristieken beschreven die geordend zijn van sterk naar zwak. Sterke heuristieken geven informatie over grote delen van een grafiek, zoals het gebruik van kwalitatief redeneren over het oneindig gedrag van een functie of bij het optellen en vermenigvuldigen van twee sub grafieken. Zwakke heuristieken geven alleen lokale informatie over de grafiek, zoals bij het berekenen van een punt van de grafiek. In hoofdstuk 2 staan twee onderzoeksvragen centraal: Beschrijft het framework de strategieën van experts bij het schetsen van formules adequaat en discriminerend? Welke strategieën gebruiken de experts bij het schetsen van formules?

In deze casestudie participeerden vijf experts, wiskundigen die betrokken zijn bij universitair wiskundeonderwijs, nationale examens, en/of schoolboeken, en drie wiskundedocenten uit het voortgezet onderwijs, die allen met hardop denken de grafiek van een complexere functie ($y = 2x\sqrt{8-x} - 2x$) moesten schetsen en een formule vinden die bij een gegeven grafiek zou kunnen passen. De video-opnamen werden getranscribeerd, opgedeeld in fragmenten die cruciale stappen van de uitleg bevatten en geanalyseerd. De resultaten toonden dat al deze stappen konden worden gecodeerd binnen het tweedimensionale framework. Zo ontstonden paden in het framework die het oplossingsproces beschreven. We concludeerden dat het framework discriminerend was, omdat verschillende strategieën van de deelnemers resulteerden in verschillende paden in het framework. De experts gebruikten verschillende strategieën bij het schetsen van de formule: sommigen richtten zich op hun repertoire van formules die ze direct konden visualiseren door middel van een grafiek; anderen vertrouwden op sterke heuristieken, zoals kwalitatief redeneren. De belangrijkste expert strategieën waren: het herkennen van functie families en het gebruik van hun prototypische grafieken, het herkennen van belangrijke kenmerken van de grafiek, het gebruik van kwalitatief redeneren bij het exploreren van delen van de grafiek, bijvoorbeeld bij het oneindig gedrag of bij het samenstellen van twee sub grafieken nadat de formule ontleed was in twee sub-formules. De docenten maakten nauwelijks gebruik van functie families en gebruikten vaker zwakkere heuristieken. Geconcludeerd werd dat expertise in het schetsen van formules niet gerelateerd kon worden aan het berekenen van afgeleiden, omdat al onze experts leken te aarzelen om dergelijke berekeningen te starten en ook fouten maakten toen ze dat probeerden.

In hoofdstuk 3 rapporteren we over het onderzoek naar de herkenningsprocessen van experts bij het schetsen van formules. In hoofdstuk 2 vonden we dat een van de expert strategieën was ‘het herkennen van een functie familie, gevolgd door het gebruik van een prototypische grafiek’. In dit hoofdstuk richtten we ons op deze direct schetsbare formules (IGF, instant graphable formulas). Een IGF is een formule die door een persoon direct gevisualiseerd kan worden door een grafiek. IGF's kunnen gezien worden als bouwstenen in het denken en redeneren met en over formules en grafieken. Deze bouwstenen kunnen worden gecombineerd (optellen, vermenigvuldigen, koppelen, enz.) tot nieuwe en meer complexe bouwstenen (bijvoorbeeld $y = -x^4$ en $y = 6x^2$ kunnen worden gecombineerd tot een 4^e graads polynoomfunctie $y = -x^4 + 6x^2$). De onderzoeksvragen in dit hoofdstuk waren: Kunnen we het repertoire van IGF’s van experts beschrijven met behulp van categorieën van functie families? Waar letten experts op bij het koppelen van formules aan grafieken van IGF's, beschreven in termen van prototypes en kenmerken? De vijf experts van hoofdstuk 2 werkten aan een kaart-sorteertaak om te onderzoeken welke functie families de experts gebruiken, aan een matching taak om de herkenning te onderzoeken, en aan een meerkeuzetaak, met hardop denken, om de herkenningsprocessen van experts in beeld te brengen. De resultaten van de experts in de kaart-sorteringstaak toonden dat de categorieën die ze construeerden, en de categoriebeschrijvingen, sterk op elkaar leken, hoewel sommige experts meer subcategorieën maakten (bijv. verschillen tussen bergparabolen en dalparabolen). Deze categorieën waren nauw verwant aan de basisfunctie families die in het voortgezet onderwijs worden onderwezen: lineaire functies, polynoomfuncties, exponentiële functies, logaritmische functies, gebroken functies, en machtsfuncties. De experts hadden geen problemen met de matching taak, waarbij ze formules moesten koppelen aan één van de 21 alternatieve grafieken.

De analyses van de hardop denkprotocollen van de meerkeuzetaak waren gebaseerd op zowel Barsalou's model van hiërarchische georganiseerde kennis (Barselou, 1992) als op Schwarz en Hershkowitz (1999) beschrijvingen van concept images met behulp van prototypen van functie families, het herkennen van kenmerken van een functie over verschillende representaties heen, en het herkennen dat verschillende formules en/of verschillende grafieken betrekking hebben op dezelfde functie. De analyses suggereerden dat de herkenning van IGF's door experts kan worden beschreven met het Barselou-model waarin formules, functie families, prototypes, een set van kenmerken, en grafieken in rijke hiërarchische netwerken met elkaar verbonden zijn.

In hoofdstuk 4 wordt onderzocht hoe het schetsen van formules met herkennen en redeneren kan worden onderwezen met als doel het inzicht van studenten in algebraïsche formules te bevorderen. De onderzoeksvraag was: Hoe kan het inzicht in algebraïsche formules van VWO 5 leerlingen door middel van het schetsen van formules worden bevorderd? In een interventie van vijf lessen van 90 minuten werden 21 VWO 5 leerlingen onderwezen in het schetsen van formules met de hand. Het ontwerp van de interventie was gebaseerd op de expert strategieën, dat wil zeggen, op basis van een combinatie van herkennen en kwalitatief redeneren. We gebruikten de principes van het onderwijzen van complexe vaardigheden, zijnde het gebruik van een hele taak benadering, met ondersteuning en reflectie (Kirschner & Van Merriënboer, 2008; Merrill, 2013; Van Merriënboer et al., 2002). Daarnaast werd ook de meta-heuristiek 'bevragen van de formule' gebruikt (Landa, 1983; Pierce & Stacey, 2007). De interventie bestond uit vijf hele taken die overeenkwamen met de niveaus van herkenning van het tweedimensionale framework. Als eerste werd aandacht besteed aan een repertoire van basisfunctie families met hun kenmerken. Vervolgens kwamen de transformaties van prototypes van de functie families aan bod. In de derde hele taak kwam het ontleden van een formule in twee sub formules en het samenstellen van de sub grafieken aan bod. In de vierde hele taak lag de focus op het herkennen van grafische kenmerken, zoals het aflezen van nulpunten en extreme waarden uit een formule. In de laatste hele taak oefenden de leerlingen expliciet het kwalitatief redeneren over oneindig gedrag, zwakkere en sterkere componenten van een formule, stijgende en dalende functies, enz. Kwalitatief redeneren wordt vaak gebruikt door experts en wordt gekenmerkt door een focus op de globale vorm van de grafiek, globale beschrijvingen, en het negeren van wat niet relevant is in de probleemsituatie.

De leerlingen maakten een schriftelijke pre-, post-test, en na vier maanden een retentietest. Alle testen bevatten een taak waarin formules geschetst moesten worden en een matching taak, die vergelijkbaar was met de taak uit hoofdstuk 3. Zes leerlingen werd gevraagd om tijdens de schetsen van formules in de pre- en posttest hardop te denken. Uit de resultaten van de pre-test bleek dat de leerlingen weinig inzicht hadden in de formules, en de hardop denkprotocollen suggereerden een gebrek aan herkennen en redeneren. De resultaten van de post-test toonden aan dat de leerlingen hun herkenning van functie families en kenmerken en hun kwalitatief redeneren verbeterd hadden. In een vragenlijst na de post-test gaven de leerlingen zelf aan dat ze algebraïsche formules beter waren gaan begrijpen. In de retentietest waren de scores op de taak met formule schetsen en de op matching taak, zoals verwacht, lager dan in de post-test, maar nog wel significant hoger dan in de pre-test. Dit suggereerde een langdurig effect van de interventie. De bevindingen van dit onderzoek suggereerden dat, hoewel veel leerlingen nog steeds problemen hadden met complexere formules, het onderwijzen van het schetsen van formules via herkennen en kwalitatief redeneren, een middel zou kunnen zijn om het inzicht in algebraïsche formules van VWO 5 leerlingen op een systematische manier te bevorderen.

Hoofdstuk 5 beschrijft het onderzoek naar de relatie tussen de vaardigheden van leerlingen in het schetsen van formules en het gebruik van symbol sense bij het oplossen van niet-standaard algebra problemen, zoals: Hoeveel oplossingen heeft deze vergelijking? Welke *y*-waarden kan deze formule hebben? Om dit soort problemen op te lossen zouden leerlingen hun vaardigheden in het schetsen van formules kunnen gebruiken, maar ook andere aspecten van de symbol sense, zoals het opgeven van de symbolische representatie en, in plaats van te starten met berekeningen, grafieken en/of redeneringen gebruiken. Dus, symbol sense die bij het schetsen van formules gebruikt kan worden is een deelverzameling van symbol sense die bij het oplossen van niet-standaard algebra problemen gebruikt kan worden. We hebben onderzocht of leerlingen de symbol sense die ze gebruikten bij het schetsen van formules ook bij andere niet-standaard algebra problemen gebruikten. De hoofdonderzoeksvraag in hoofdstuk 5 was: Wat is de relatie tussen de vaardigheden van VWO 6 leerlingen om formules te schetsen en hun gebruik van symbol sense bij het oplossen van niet-standaard algebra taken? Er werden twee deelvragen geformuleerd: In welke mate zijn de vaardigheden van leerlingen om formules te schetsen positief gecorreleerd met hun vaardigheden om algebra taken met symbol sense op te lossen? En: Is de symbol sense die leerlingen gebruiken bij het schetsen van formules vergelijkbaar met of verschillend van hun symbol sense gebruik bij het oplossen van niet-standaard algebra taken? Een groep van 114 VWO 6 leerlingen, waaronder 21 leerlingen die hadden deelgenomen aan de interventie zoals beschreven in hoofdstuk 4 en 93 leerlingen van vijf andere scholen verspreid over Nederland, maakten een schriftelijke symbol sense test. Dezelfde zes studenten die betrokken waren bij de interventie, werden gevraagd om hardop te denken tijdens deze symbol sense test, die bestond uit 8 schets taken en 12 niet-standaard algebra taken. De resultaten van de schriftelijke test werden gescoord en het symbol sense gebruik werd geanalyseerd en gescoord aan de hand van vier categorieën: blanco, berekeningen, het maken van een grafiek, herkennen en redeneren.

Er werden positieve correlaties gevonden tussen de vaardigheden van de leerlingen om formules te schetsen en hun vaardigheden om de algebra taken op te lossen, en ook met hun gebruik van symbol sense bij het oplossen van deze taken. Ook wanneer gecorrigeerd werd voor de algemene wiskundige vaardigheden van de leerlingen veranderden deze correlaties nauwelijks. Leerlingen die hoog scoorden op het schetsen van grafieken maakten vaker gebruik van de strategie 'een grafiek maken' bij het werken aan de algebra taken. Met betrekking tot de tweede sub vraag vonden we dat 16 van de 21 leerlingen die waren betrokken bij de interventie uit hoofdstuk 4, behoorden tot de 25% hoogst scorende leerlingen op de formule-schets-taken. Deze hoog scorende leerlingen gebruikten meer symbol sense bij het oplossen van de niet-standaard algebra taken dan de andere studenten. De zes leerlingen die hardop dachten behoorden allen tot de groep van 25% hoogst scorende leerlingen. Zij gebruikten vergelijkbare aspecten van symbol sense bij zowel het schetsen van formules als bij het oplossen van niet-standaard algebra taken, zoals het herkennen van functie families en kenmerken, en kwalitatief redeneren. Deze bevindingen bevestigden onze verwachtingen dat leerlingen die in staat zijn om formules te schetsen, deze vaardigheden kunnen gebruiken in een breder domein van niet-standaard algebra taken.

In het afsluitende hoofdstuk 6 presenteren we eerst de bevindingen van de vier afzonderlijke studies, gevolgd door discussie en conclusies met de belangrijkste bijdragen en beperkingen van de studie. Het belangrijkste doel van dit onderzoek was het bevorderen van aspecten van de symbol sense van leerlingen, dat wil zeggen het vermogen van leerlingen om formules te doorzien, betekenis te kunnen geven aan formules, en deze symbol sense te gebruiken bij het oplossen van algebra problemen. Het uitgangspunt van ons onderzoek was dat leerlingen betekenis moeten kunnen geven aan algebraïsche formules en dat ze daarom deze formules moeten kunnen doorzien. Immers, als leerlingen geen betekenis kunnen geven aan algebra dan zullen ze geen vertrouwen ontwikkelen in hun algebraïsche werk, hetgeen zou kunnen leiden tot een terughoudendheid in het algebraïsch redeneren en denken. Om expertise en vertrouwen in het lezen van algebraïsche formules te laten ontwikkelen, hebben we gekozen voor een klein maar rijk domein in algebra, namelijk de schetsen van formules. Het schetsen van formules vereist dat leerlingen vele formules moeten kunnen doorzien en stelt ze in staat om betekenis te geven aan deze formules door ze te koppelen aan een grafiek. Omdat we het inzicht in formules wilden bevorderen, hebben we ons beperkt tot het interpreteren van formules en zijn we voorbijgegaan aan algebraïsche manipulaties, die vaak de kern vormen van het reguliere algebra-onderwijs en een bron van problemen is voor veel studenten. We kozen ervoor om formules met de hand te schetsen, omdat het koppelen van een formule aan een grafiek met de hand effectiever lijkt dan via technologie. Experts worden verondersteld inzicht in formules te hebben en daarom onderzochten we hoe experts formules schetsen. We identificeerden essentiële denkprocessen van experts en beschreven deze met een tweedimensionaal framework in termen van herkennen en redeneren. Dit gaf ons aanwijzingen over wat te onderwijzen aan leerlingen. Op basis van het tweedimensionale framework ontwierpen we een interventie van vijf lessen van 90 minuten, het zogenaamde GQR-ontwerp (Graphing based on Qualitative reasoning and Recognition). Door middel van hele taken, met ondersteuning via hulpvragen en met reflectievragen, en met behulp van 'het bevragen van de formule' als een leidende meta-heuristiek, werd het schetsen van formules stap voor stap en op een systematische manier onderwezen. In dit GQR-ontwerp wordt expliciet aandacht besteed aan het samenspel tussen herkennen en redeneren via het gebruik van functie families met hun prototypische grafieken als bouwstenen, kenmerken van formules en kwalitatief redeneren. De hele taakbenadering stimuleert leerlingen tot globaal kijken ten behoeve van herkenning, tot redeneren en argumenteren, en tot het overwegen van hun strategieën, hetgeen essentiële aspecten van symbol sense zijn (Drijvers et al., 2011). We verwachtten dat de leerlingen deze aspecten van symbol sense ook konden gebruiken in andere situaties zoals bij het oplossen van niet-standaard algebra problemen. Daarvoor ontwierpen we een symbol sense test met grafiek-schets-taken en niet-standaard algebra taken, die konden worden opgelost met herkennen, redeneren en het maken van een grafiek. De resultaten van deze symbol sense test suggereerden dat de leerlingen die betrokken waren bij de interventie in staat waren om hun symbol sense geleerd bij het schetsen van formules konden gebruiken bij het oplossen van de algebra taken. We concludeerden dat het schetsen van formules onderwezen met ons GQR-ontwerp een effectief middel zou kunnen zijn om leerlingen in de hogere klassen van voortgezet onderwijs aspecten van symbol sense te leren, zoals inzicht in algebraïsche formules en kwalitatief redeneren, die gebruikt kunnen worden om niet-standaard algebra taken op te lossen.

De gerapporteerde studies hebben onvermijdelijk hun beperkingen. In de hoofdstukken 2 en 3 namen slechts vijf experts deel en werkten ze slechts aan twee taken, vanwege de arbeidsintensieve onderzoeksmethode. Hoewel we verwachten dat de meeste strategieën in het tweedimensionale framework zijn vastgelegd, zou het testen van een grotere groep potentiële experts een ander, gedetailleerder beeld van expertise kunnen geven. Bij de interventie in hoofdstuk 4 was slechts één klas Nederlandse leerlingen betrokken en was er geen controlegroep. Een jaar later en twee jaar later werd echter dezelfde reeks lessen uit de interventie gebruikt in twee andere klassen van dezelfde school, beide met 23 leerlingen. Beide groepen maakten dezelfde post-test die in hoofdstuk 4 werd gebruikt, en de scores lieten vergelijkbare resultaten zien. Hoewel dit een bevestiging kan zijn dat leerlingen inzicht in formules kunnen ontwikkelen via ons GQR-ontwerp, zouden in de toekomst meer leerlingen en docenten betrokken kunnen worden bij verder onderzoek naar het GQR-ontwerp.

In de symbol sense test in hoofdstuk 5 hebben we een combinatie van taken met het schetsen van formules en algebra taken gebruikt voor onderzoeksdoeleinden. Dit zou leerlingen de suggestie hebben kunnen geven om bij het werken aan de algebra taken gebruik te maken van de strategie ’het maken van een grafiek’. In de symbol sense test gebruikten we vaak de variabelen x en y om de test herkenbaar te maken voor leerlingen. In toekomstige onderzoeken stellen we voor om vaker andere variabelen dan x en y te gebruiken. De algebra taken in de huidige test waren beperkt tot de taken die met behulp van grafieken en redeneringen konden worden opgelost. In toekomstige testen zouden ook andere algebra problemen kunnen bevatten. Algebra problemen vereisen vaak een combinatie van redeneringen, grafieken en berekeningen, en een volgende stap zou kunnen zijn om ook taken op te nemen waarin men moet overwegen of berekeningen nodig zijn of niet, zoals bij ‘hoeveel oplossingen heeft de vergelijking $3.6(1 - \mathrm{e}^{-2.5t}) = 10t$?’ In de symbol sense test vonden we dat veel leerlingen problemen hadden met het oplossen van ongelijkheden zoals $x(x-1) > 4x$. In plaats van hun vaardigheden in het schetsen van formules te gebruiken, begon de helft van de groep van 25% hoogst scorende leerlingen bij de formule-schets-taken te rekenen, hetgeen vrijwel nooit succesvol was. De bevindingen bij het hardop denken leken te suggereren dat een ongelijkheid eerder geleerde associaties oproept, en dat dergelijke associaties later geleerde symbol sense in de weg zouden kunnen staan. Verder onderzoek is nodig om te onderzoeken hoe net geleerde symbol sense kan worden opgenomen in de strategieën en gewoontes van de studenten om met algebra problemen om te gaan. In hoofdstuk 5 suggereerden we dat het schetsen van formules op basis van herkennen en redeneren een middel zouden kunnen zijn om symbol sense te onderwijzen in de bovenbouw van het voortgezet onderwijs, en dat deze symbol sense gebruikt zou kunnen worden om niet-standaard algebra taken op te lossen. Meer onderzoek is nodig om deze suggestie te onderbouwen. Een volgende stap zou kunnen zijn om een quasi-experimentele studie met controlegroep en met een pre- en post-test op te zetten, waarbij een groep leerlingen wordt onderwezen om formules te schetsen, zoals in de interventie. Omdat we verwachten dat de problematiek met inzicht in algebraïsche formules en symbol sense niet exclusief is voor Nederlandse leerlingen, zouden ook leerlingen en docenten uit andere landen in toekomstige studies moeten worden opgenomen.

Dit proefschrift draagt bij aan onze kennis over symbol sense en het onderwijzen van symbol sense. Ten eerste beschrijft het de aard van de expertise in termen van een samenspel tussen herkennen en redeneren, en het laat zien hoe dit kan worden uitgewerkt voor het domein van het schetsen van formules. Ten tweede laat het zien hoe leerlingen in de bovenbouw van het voortgezet onderwijs inzicht in algebraïsche formules kunnen krijgen door middel van een innovatieve lessenserie over het schetsen van formules. En ten derde wordt onderzocht hoe symbol sense om niet-standaard algebra problemen op te lossen onderwezen kan worden.

De eerste bijdrage van dit onderzoek is de beschrijving van de aard van expertise in het schetsen van formules met een tweedimensionaal framework in termen van herkennen en heuristieken. We identificeerden verschillende niveaus van herkenning die de probleemruimte en dus het heuristisch zoeken bepalen: de herkenning stuurt de heuristische zoektocht. Op verschillende niveaus van herkenning hebben we in het tweedimensionale framework heuristieken geformuleerd, waarbij we de nadruk legden op het samenspel tussen herkennen en domein specifieke heuristieken. Deze benadering wijkt af van bijvoorbeeld beschrijvingen van kennisbases, waarin wiskundige competenties worden beschreven in lijsten met verschillende componenten, zoals conceptuele kennis, procedurele kennis en strategische competentie. Hoewel de noodzaak van integratie van verschillende componenten vaak is benadrukt (Kilpatrick, Swafford, & Findell, 2001), heeft dit, bij ons weten, niet geleid tot modellen waarin deze componenten daadwerkelijk zijn geïntegreerd. Het beschrijven van expertise in termen van herkennen en heuristieken in een tweedimensionaal framework, lijkt ook mogelijk in andere domeinen van algebra. Pouwelse, Janssen en Kop (ingediend voor publicatie) stelden een vergelijkbaar framework op voor het vinden van onbepaalde integralen in calculus. Een dergelijk framework kan gebruikt worden als instrument voor het ontwerpen van lesmateriaal, maar ook als instrument in de professionele ontwikkeling van docenten. Verder onderzoek is nodig om te onderzoeken hoe het samenspel tussen herkennen en domein specifieke heuristieken in andere domeinen zoals het oplossen van vergelijkingen kan worden beschreven en gebruikt bij het ontwerpen van onderwijs en/of bij de professionele ontwikkeling van docenten.

Als tweede bijdrage van het onderzoek hebben we laten zien hoe leerlingen in de bovenbouw inzicht kunnen krijgen in algebraïsche formules door middel van het schetsen van formules via ons GQR-ontwerp. Het GQR-ontwerp verschilt van zowel de reguliere als van andere innovatieve benaderingen van het onderwijzen over functies, met name wat betreft de koppeling tussen formules en grafieken. Het reguliere onderwijs over functies richt zich vaak op het manipuleren van algebraïsche expressies (Arcavi et al., 2017; Schwartz & Yerushalmy, 1992) en op het gebruik van technologie, zoals de grafische rekenmachine, om functie families te verkennen en om een 'plaatje' te maken van een functie. In vergelijking met reguliere benaderingen wordt in het GQR-ontwerp expliciet aandacht besteed aan herkennen en aan redeneren met en over functies. In ons ontwerp gebruiken we basisfunctie families als bouwstenen voor formules, in navolging van Davis' suggestie om grotere denkeenheden te gebruiken om de structuur van de formule beter te kunnen herkennen (Davis, 1983), en we besteden aandacht aan het aflezen van belangrijke kenmerken uit de formules. Bovendien wordt er expliciet aandacht besteed aan het redeneren, bijvoorbeeld over parameters van functie families, over oneindig gedrag, en bij het optellen en vermenigvuldigen van twee sub grafieken. Andere innovatieve benaderingen richten zich vaak op het redeneren over functies met behulp van technologie. In vergelijking met onze aanpak besteden deze benaderingen geen expliciete aandacht aan kwalitatief redeneren en aan het herkennen en gebruiken van functie families. Het belang van kwalitatief redeneren en het ontbreken daarvan in de wiskundecurricula werd al gesignaleerd door Leinhardt et al. (1990), Goldenberg et al. (1992), Yerushalmy (1997), en Duval (2006), maar voor zover wij weten is dit kwalitatief redeneren nog nooit geïmplementeerd in concrete en systematische onderwijsbenaderingen.

De derde bijdrage van dit onderzoek is dat het laat zien hoe symbol sense om niet-standaard algebra problemen op te lossen aan leerlingen zou kunnen worden onderwezen.

Symbol sense lijkt moeilijk te onderwijzen (Arcavi et al., 2017; Hoch & Dreyfus, 2005). In het reguliere wiskundeonderwijs in het voortgezet onderwijs richten veel docenten en leerlingen zich op basisvaardigheden met het manipuleren van algebraïsche expressies (Arcavi et al., 2017), in de verwachting dat leerlingen door dit soort oefening symbol sense zullen ontwikkelen. Innovatieve benaderingen richten zich meer op het redeneren en geven suggesties hoe dit te onderwijzen, bijvoorbeeld door gebruik te maken van productieve oefeningen, zoals omgekeerd denken en het construeren van eigen voorbeelden (Friedlander & Arcavi, 2012; Kindt, 2011), door gebruik te maken van rijke, collaboratieve taken (Swan, 2008), en door snapshots voor discussies in de klas (Pierce & Stacey, 2007). De kern van onze aanpak is het idee om symbol sense te onderwijzen in een beperkt domein van algebra, het schetsen van formules. Op deze wijze zouden leerlingen expertise in zo'n beperkt domein kunnen ontwikkelen. Als het onderwijzen van het schetsen van formules zich richt op essentiële aspecten van symbol sense, zoals globaal kijken ten behoeve van herkenning, (kwalitatief) redeneren en strategisch werken, dan zouden deze essentiële aspecten van symbol sense ook gebruikt kunnen worden in een breder domein van algebra. In ons GQR-ontwerp werden deze essentiële aspecten van symbol sense expliciet en systematisch onderwezen als denkgereedschap, terwijl dit soort denkgereedschap in andere benaderingen vaak impliciet blijft. Ons onderzoek toont aan dat leerlingen inzicht kregen in formules, en essentiële aspecten van symbol sense leerden, die ze konden gebruiken bij het oplossen van niet-standaard algebra problemen. De leerlingen die bij de interventie betrokken waren, gaven aan dat ze dachten dat ze functies beter waren gaan begrijpen, dat ze formules beter konden visualiseren, met name basisfuncties, en bovendien dat kwalitatief redeneren voor hen heel nieuw en motiverend was ("we gebruiken nu globaal redeneren; het is leuk, dit soort redeneren"). Dit suggereert dat het GQR-ontwerp een motiverende en systematische manier kan zijn om leerlingen aspecten van symbol sense te onderwijzen.

Het vermogen om formules te doorzien en er betekenis aan te geven is een belangrijk aspect van symbol sense en zal ook in de toekomst belangrijk blijven wanneer technologie het manipuleren van algebraïsche formules nog verder zal overnemen. Mensen zullen wel de resultaten hiervan moeten interpreteren, globale schattingen moeten maken van de resultaten en moeten begrijpen wat er aan de hand is. Daarvoor zullen ze een soort formule sense moeten ontwikkelen, waarmee ze betekenis kunnen geven aan formules, functie families als bouwstenen van formules kunnen gebruiken, de structuur van formules kunnen identificeren, en kwalitatief redeneren kunnen gebruiken. Deze ideeën kunnen interessant zijn voor het ontwikkelen van nieuwe curricula voor het voortgezet onderwijs. In dit proefschrift hebben we met ons GQR-ontwerp laten zien hoe deze symbol sense kan worden onderwezen aan leerlingen.

CURRICULUM VITAE

Peter Kop was born in The Hague, in the Netherlands, on 28 August 1954. He attended the Maria Mulo and Bonaventura College in Leiden and studied mathematics, economics and education at Leiden University from 1974 to 1978. He started his teaching career at the St. Gregoriuscollege in Utrecht. In this period, he participated in writing a math textbook “Wiskunde doe het zelf” (Math do it yourself). From 1990 onwards, he was for over 20 years involved in the production of national math A exams. For some years he was on the board of the NVvW (Dutch Society for Math Teachers) and was member, and subsequently chairman, of the editorial board of the ‘Zebra-reeks’, a series of textbooks focusing on mathematics and applications outside the curriculum, presenting interesting aspects of maths for students in upper secondary school. In 1999 he moved to GSG LeoVroman in Gouda, and in 2002 he started as ‘vakdidacticus’ (math teacher educator) at ICLON, Graduate School of Teaching at Leiden University. He was involved in several projects on national math curriculum reforms and participated in the writing of several chapters of the Handboek Wiskundedidactiek. In 2011 he started on a PhD on algebra education. His main interests are teaching meaningful algebra and statistics challenging students’ common sense, translating teaching ideas and research into teaching materials, and mathematical thinking and teaching it.

PUBLICATIONS

Kop, P. M., Janssen, F. J., Drijvers, P. H., & van Driel, J. H. (2020b). The relation between graphing formulas by hand and students' symbol sense. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-020-09970-3

Kop, P. M., Janssen, F. J., Drijvers, P. H., & van Driel, J. H. (2020a). Promoting insight into algebraic formulas through graphing by hand. *Mathematical Thinking and Learning*, 1-20. https://doi.org/10.1080/10986065.2020.1765078

Kop, P. M., Janssen, F. J., Drijvers, P. H., & van Driel, J. H. (2017). Graphing formulas: Unraveling experts' recognition processes. *The Journal of Mathematical Behavior*, *45*, 167–182.

Kop, P. M., Janssen, F. J., Drijvers, P. H., Veenman, M. V., & van Driel, J. H. (2015). Identifying a framework for graphing formulas from expert strategies. *The Journal of Mathematical Behavior*, *39*, 121–134.

Kop, P., Janssen, F., Drijvers P., & van Driel, J. (2019). Teaching graphing formulas by hand as a means to promote students' symbol sense. In Graven, M., Venkat, H., Essien, A. & Vale, P. (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol 2, pp. 488 –495). Pretoria, South Africa: PME.

Kop, P., Janssen, F., Drijvers P., & van Driel, J. (2019). Graphing formulas to give meaning to algebraic formulas. In Jankvist, U. T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.). *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 582–589). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Kop, P.M.G.M. (2017, July). Teaching the graphing of formulas to improve students' symbol sense. *Proceedings IMA and CETL-MSOR: Mathematics Education beyond 16: Pathways and Transitions*. https://ima.org.uk/2996/mathematics-education-beyond-16-pathways-transitions/.

Kop, P., Janssen, F., Drijvers, P., van Driel, J., & Veenman, M. (2014). Identifying a framework for graphing formulas from expert strategies. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.). *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education* (Vol. 6, p.136). Vancouver, Canada: PME

Kop, P., Janssen, F., & Drijvers, P. (2013). How experts graph formulas. In B. Ubuz, C. Haser, & M.A. Mariotti (Eds.). *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 568–569). Ankara, Turkey: Middle East Technical University and ERME.

Presentations

Kop., P. (2020, March). *Hoe symbol sense te onderwijzen?* Presentation at Flos wiskundesymposium, Sittard.

Kop, P. (2019, February). *Teaching graphing formulas by hand to improve students'symbol sense*. Presentation at Nationale Wiskunde Dagen Veldhoven.

Barneveld., E. van, & Kop, P. (2019, February). *Betekenis geven aan algebraïsche formules.* Presentation at Nationale Wiskunde Dagen. Veldhoven.

Kop, P.M.G.M., Janssen, F., Drijvers, P., & van Driel, J. (2018). *Graphing formulas with pen and paper by experts (and students)*. Poster presentation at NVvW conferentie Onderwijs meets Onderzoek, Utrecht.

Kop, P.M.G.M., Janssen, F., Drijvers, P., & van Driel, J. (2017, June). *Can graphing formulas by hand improve students' symbol sense?* Poster presentation at NVvW conferentie Onderwijs meets Onderzoek, Utrecht.

Kop, P.M.G.M. (2017, May). *Recognition and heuristic search in graphing formulas (with paper and pen).* Presentation at Scholennetwerk Amsterdam, Amsterdam.

Kop, P.M.G.M. (2017, February). *Grafieken schetsen: een (gemiste?) algebraïsche vaardigheid*. Presentation at Nationale Wiskunde Dagen, Noordwijkerhout.

Kop, P.M.G.M. (2016, December). *Grafieken schetsen: een (gemiste?) algebraïsche vaardigheid*. Presentation at Nascholing Wiskunde "De vele gezichten van wiskunde". Universiteit Groningen.

Kop, P.M.G.M., Janssen, F., Drijvers, P., & van Driel, J. (2016, July). *Recognition and heuristics in graphing formulas.* Presentation at *International Congress on Mathematical Education: ICME–13*. Hamburg, Germany.

Kop, P.M.G.M., Janssen, F., Drijvers, P., & van Driel, J. (2016, June). *Graphing formulas with pen and paper by experts (and students).* Poster presentation at NVvW conferentie Onderwijs meets Onderzoek, Utrecht.

Other publications

Kop, P., Oord, R. van, Barneveld, E. van, & Voorhoeve, M. (2019). Wiskundige denkactiviteiten bij optimaliseren. *Euclides, 94*(4).

Kop, P., Voorhoeve, M., & Streun, A. van (2018). Wiskunde. In F. Janssen, & H. Hulshof, & K. van Veen (Eds.), *Wat is echt de moeite waard om te onderwijzen* (pp. 289–312). ICLON Universiteit Leiden, ILO Rijksuniversiteit Groningen.

Kop, P. (2018). Statistiekonderwijs in de onderbouw voor vandaag en morgen. In G. Bruin-Muurling, D. van Eerde, F. Van Galen, K. Gravemeijer, I. and Van Stiphout (Eds.), *Statistiekonderwijs voor Morgen* (pp. 39–50). Utrecht: Wiskunde voor Morgen.

Wesker, L., & Kop, P. (2018). Met wijsheid en mildheid. *Euclides, 95*(5).

Streun, A. van, & Kop, P. (2017). *Ontwerpen van wiskundige denkactiviteiten onderbouw havo-vwo.* Implementatie examenprogramma havo-vwo 2015. SLO publicaties, Enschede.

Soto y Koelemeijer, G. S., Janssen, F. J. J. M., & Kop, P. M. G. M. (2017). Shaping Mathematics into Stories by means of Propp's Narratemes. *EURASIA Journal of Mathematics, Science and Technology Education*, *14*(2), 657-669.

Drijvers, P., & Kop, P. (2016). Variabelen en vergelijkingen. In P. Drijvers, A. van Streun, & G. Zwaneveld (Eds.), *Handboek wiskundedidactiek* (pp. 53–81) Epsilon Uitgaven, Utrecht.

Kop, P., & Hoekstra, W. (2016). Functies. In P. Drijvers, A. van Streun, & G. Zwaneveld (Eds.), *Handboek wiskundedidactiek* (pp. 83–109). Epsilon Uitgaven, Utrecht.

Van Streun., A. & Kop, P. (2016). Wiskundge denkactiviteiten. In P. Drijvers, A. van Streun, & G. Zwaneveld (Eds.), *Handboek wiskundedidactiek* (pp. 339–368). Epsilon Uitgaven, Utrecht.

Kop, P., & Streun, A. van (2016). Wiskunde. In: F. Janssen, H. Hulshof, K. van Veen (Eds.), *Uitdagend gedifferentieerd vakonderwijs* (pp. 283–306). ICLON Universiteit Leiden, ILO Rijksuniversiteit Groningen.

Streun, A. van, & Kop, P. (2016). Ontwerpen van wiskundige denkactiviteiten bovenbouw havo-vwo Implementatie examenprogramma havo-vwo 2015. SLO publicaties, Enschede.

Daemen, J., Limpens, G., Kop, P., & Voorhoeve, M. (2015). Wiskunde wandeling. *Euclides, 91*(1).

Barneveld, E. van, Boer, W., Giessen, C. van de, Kop, P., Ree, H. van der, Reuling, H., Spijkers, F., Stroosma, T., & Verschut, A. (2015). Modules wiA havo domein statistiek : ontwikkeld in het kader van de nieuwe examenprogramma's voorgesteld door de commissie Toekomst Wiskunde Onderwijs (Ctwo) en herzien door SLO. https://nvvw.nl/vakinhoudelijk/statistiek/

Broek, L., & Kop, P. (2009). *Experimenteel lesmateriaal voor verbanden.* https://www.fi.uu.nl/ctwo/.

Broek, L., & Kop, P. (2009). *Experimenteel lesmateriaal voor veranderingen.* https://www.fi.uu.nl/ctwo/.

Other presentations

Kop, P., & Wesker, L. (2018, November). *Mild en rechtvaardig wiskunde B examens corrigeren.* Presentation at Studiedag NVvW.

Gravemeijer, K., Galen, F. van, & Kop, P. (2018, November). *Statistiek voor morgen.* Presentation at Studiedag NVvW.

Kop, P.M.G.M. (2017, November). *Wiskundige denkactiviteiten in de klas.* Presentation at NUWiskundecongres, Nieuwegein.

Peter Kop en Hester Vogels (2017, November). *Vernieuwe statistiek programma.* Presentation at Studiedag NVvW

P.M.G.M. Kop (2017, June). *Vernieuwde statistiek havo! En vwo?* Presentation at Wiskundedialoog, Nijmegen.

P.M.G.M. Kop (2017, March). *A categorization of equations from expert-students.* Presentation at GDM 2017 in Sektion 6: Erkennen von Strukturen im Kontext algebraischen Denkens, by Janßen, T., Block, J., Kop, P.

P.M.G.M. Kop (2016, June). *Vernieuwde statistiek havo. En vwo?* Presentation at NVvW Conferentie, Utrecht.

P. Kop, E. van Barneveld, I. de Jong (2015, November). *Nieuwe statistiek in wiskunde A.* Presentation at Studiedag NVvW.

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Verhoeven, P. (1997). *Tekstbegrip in het onderwijs klassieke talen.*

Meijer, P.C. (1999). *Teachers' practical knowledge: Teaching reading comprehension in secondary education.*

Zanting, A. (2001). *Mining the mentor's mind: The elicitation of mentor teachers' practical knowledge by prospective teachers.*

Uhlenbeck, A.M. (2002). *The development of an assessment procedure for beginning teachers of English as a foreign language.*

Oolbekkink-Marchand, H.W. (2006). *Teachers' perspectives on self-regulated learning: An exploratory study in secondary and university education.*

Henze-Rietveld, F.A. (2006). *Science teachers' knowledge development in the context of educational innovation.*

Mansvelder-Longayroux, D.D. (2006). *The learning portfolio as a tool for stimulating reflection by student teachers.*

Meirink, J.A. (2007). *Individual teacher learning in a context of collaboration in teams.*

Nijveldt, M.J. (2008). *Validity in teacher assessment: An exploration of the judgement processes of assessors.*

Bakker, M.E.J. (2008). *Design and evaluation of video portfolios: Reliability, generalizability, and validity of an authentic performance assessment for teachers.*

Oonk, W. (2009). *Theory-enriched practical knowledge in mathematics teacher education*.

Visser-Wijnveen, G.J. (2009). *The research-teaching nexus in the humanities: Variations among academics.*

Van der Rijst, R.M. (2009). *The research-teaching nexus in the sciences: Scientific research dispositions and teaching practice.*

Platteel, T.L. (2010). *Knowledge development of secondary school L1 teachers on concept-context rich education in an action-research setting*.

Kessels, C.C. (2010). *The influence of induction programs on beginning teachers' well-being and professional development*.

Min-Leliveld, M.J. (2011). *Supporting medical teachers' learning: Redesigning a program using characteristics of effective instructional development.*

Dobber, M. (2011). *Collaboration in groups during teacher education*.

Wongsopawiro, D. (2012). *Examining science teachers pedagogical content knowledge in the context of a professional development program*.

Belo, N.A.H. (2013). *Engaging students in the study of physics: An investigation of physics teachers' belief systems about teaching and learning physics*.

De Jong, R.J. (2013). *Student teachers' practical knowledge, discipline strategies, and the teacher-class relationship*.

Verberg, C.P.M. (2013). *The characteristics of a negotiated assessment procedure to promote teacher learning*.

Van Kan, C.A. (2013). *Teachers' interpretations of their classroom interactions in terms of their pupils' best interest: A perspective from continental European pedagogy*.

Dam, M. (2014). *Making educational reforms practical for teachers: Using a modular, success-oriented approach to make a context-based educational reform practical for implementation in Dutch biology education.*

Hu, Y. (2014). *The role of research in university teaching: A comparison of Chinese and Dutch teachers.*

Vink, C.C. (2014). *Mapping for meaning: Using concept maps to integrate clinical and basic sciences in medical education*.

De Hei, M.S.A. (2016). *Collaborative learning in higher education: design, implementation and evaluation of group learning activities.*

Louws, M.L. (2016). *Professional learning: what teachers want to learn.*

Moses, I. (2017). *Student-teachers' commitment to teaching.*

Veldman, I. M.J. (2017). *Stay or leave? Veteran teachers' relationships with students and job satisfaction.*

Chen, D. (2017). *Intercultural identities of English language teachers: An exploration in China and the Netherlands.*

Vereijken, M.W.C. (2018). *Student engagement in research in medical education.*

Stollman, S.H.M. (2018). *Differentiated instruction in practice: A teacher perspective.*

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Tran, T.T.Q. (2018). *Cultural differences in Vietnam: Differences in work-related values between Western and Vietnamese culture and cultural awareness at higher education.*

Van Ginkel, G.V.M. (2018). *Making mentoring match: Mentor teachers' practical knowledge of adaptive mentoring.*

Wieringa, N. (2019). *Teacher knowledge and lesson design: Understanding and supporting biology teachers' decision-making while designing context-based lessons.*

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Van Kampen, E. (2019). *What's CLIL about bilingual education? A window on Content and Language Integrated Learning pedagogies.*

Den Ouden, J.A.J. (2020). *Zachtjes schudden aan de boom: Een onderzoek naar rationales en kernpraktijken van eerstegraads docenten Godsdienst/Levensbeschouwing gericht op de levensbeschouwelijke identiteitsontwikkeling van hun leerlingen.*

Vogelzang, J. (2020). *Scrum in secondary chemistry education: A methodology to support teachers and to scaffold students.*

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