Patsy Haccou

Mathematics
for
Global challenges

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Contents

Preface

This book gives an introduction to mathematical reasoning and modelling in the context of global challenges. It can be used as a textbook for bachelor courses at different levels, in the first and second years of college. Although part III contains most of the foundational mathematical knowledge that is needed, it is assumed that students using this book have a sound mathematical background knowledge.

The setup and layout of this book

The main text is presented in a Tufte-style layout, using a broad margin that contains (most of) the figures, and comments, such as background information, references to other sections and chapters, citations, and optional extras. The pdf version of the book includes red hyperlinks to internal references and green-coloured links to websites.

The setup of the book is modular: part I contains applications of models, part II the modelling tools, and III the mathematical foundations for these tools.

Part I of the book shows examples of the ultimate use of mathematics in practical contexts that are related to global challenges. Each chapter in this part focuses on a different context, and consists of a short introduction to the general subject, followed by one or more model sections that contain a series of assignments. By doing these assignments students are gradually introduced to one or more models, their analysis, and the implications of their main results in the considered practical context. The background information for doing the assignments in part I is provided in part II. That part begins with a chapter that gives a general introduction to modelling, followed by several chapters that each focus on a specific class of models. Part III focuses on basic mathematical concepts and techniques that underly the models in parts I and II. References to these foundational subjects are provided in the margin notes. Which of the chapters in this part are relevant for individual students depends on their background knowledge.

This book is not meant to be read in one go, from page one to the end. Rather, the way to use this book is to pick a subject of interest in part I, and work on the assignments. Each of the model sections in I should be considered as a long-term project, that will gradually increase the understanding of a specific type of models. To guide this process, references to the relevant book chapters and sections in the later parts are given in the margins.

Parts II and III also contain exercises and assignments, but of a different kind: these are meant for practice and self-testing. The answers to these exercises are provided in chapters at the end of each of these parts. The answers to the assignments in part I are not included in the student version of the textbook. By studying the background knowledge that is provided in the rest of the book, students should eventually be able to find the solutions to these assignments by themselves.

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Part I

Challenges

1

The wisdom of crowds

Written in collaboration with Anne-Mieke Thieme

What is the value of diversity? Is it a desirable quality on its own? Should we strive for diversity even if it might be a hassle to achieve? These questions are at the heart of discussions about quota. Some argue that requirements should be set for the distribution of gender, sexuality, race, ethnicity or religion at organisations. They say that the make-up of student populations at universities, employees at companies and positions at organisations should reflect the diverse society we live in. Others posit that quota are an undesirable form of positive discrimination and that recruitment should always be based on merit.

Mathematical reasoning can help to shed some light on this discussion about diversity. Under certain conditions, it turns out, diversity has an added value. In these circumstances, diverse groups of reasonably intelligent people give more accurate answers than homogeneous groups of reasonably intelligent people, and can even outperform experts. This idea was coined by James Surowiecki in his book *The Wisdom of Crowds* (2004). [1,2].

[1] J. Surowiecki. *The Wisdom of Crowds*. Doubleday, 2004

[2] S. E. Page. *The Difference: How the Power of Diversity Creates Better Groups, Firms, Schools, and Societies*. Princeton University Press, Princeton, NJ, 2007a

The reasoning behind the idea is elegantly simple. Confronted with a difficult problem, such as making predictions under uncertain conditions, it is only natural that individuals, even experts, are slightly off the correct answer. After all, to err is human, especially when the issue is complicated. If we assume that people make errors in all directions (some underestimate and others overestimate the accurate answer), individual errors are cancelled out by being aggregated in a group. Together, the crowd arrives at an accurate answer.

Diversity sometimes adds value when making decisions, but not always. One condition is that the problem should be such that all individuals are prone to making mistakes. Additionally, the problem must have a real answer, or an optimum outcome, so the level of accuracy of an answer can be measured. Moreover, the individuals in the group must all be reasonably smart and good at representing their unique perspectives. They must be sufficiently informed about the issue. The group of diverse problem solvers must be sufficiently large, and should have been drawn from an even larger population of potential problem solvers[3].

Conditions for the 'diversity prediction' theorem.

[3] S. E. Page. Making the difference: Applying a logic of diversity. *Academy of Management Perspectives*, 21:6–20, 2007b

The last condition relates to the independence of the answers given. The wisdom of the crowd can go wrong if some individuals may influence the others, for instance if there is a hierarchy in the group, if they imitate each other's answers, or if emotional factors, such as peer pressure, are involved[4]. It has been shown that the more information participants are given about other people's guesses, the more the crowd's answer deviates from the actual one [5]. This deviation might be explained by hierarchical factors, herding and social influence factors, or by the fallacy ad populum, the idea that "the opinion of the majority is the correct opinion".

An example of the wisdom of the crowd in action is the 'ask the audience' lifeline in Who Wants to be a Millionaire? In the 1337 studied cases, the audience was correct 91% of the time[6]. Companies such as Google also use the wisdom of the diverse crowd to make predictions. At Hewlett Packard, managers predict printer sales. Their averaged answer turns out to be as good as, or even better than that of experts[7].

In conclusion, it can be shown mathematically that under certain circumstances diversity is a desirable quality which leads to better outcomes. In these cases, it can be a more important factor than expertise, and it results in more effective decision-making. Therefore, striving for diversity in the composition of boards, employees and institutions is supported by mathematical theory, provided that the above mentioned conditions are satisfied.

[4] E. Vul and H. Pashler. Measuring the crowd within: Probabilistic representations within individuals. *Psychological Science*, 19:645–647, 2008

[5] J. Lorenz H. Rauhut F. Schweitzer D. Helbing. How social influence can undermine the wisdom of the crowd effect. *Proceedings of the National Academy of Science in the United States of America*, 108:9020–9025, 2011

[6] Franzen A. and S. Pointner. Calling social capital: An analysis of the determinants of success on the tv quiz show who wants to be a millionaire? *Social Networks*, 33:79–87, 2011

[7] Charles R.l Plott and Kay-Yut Chen. *Information Aggregation Mechanisms: Concept, Design and Implementation for a Sales Forecasting Problem*, volume 1131. California Institute of Technology, Pasadena, CA, 2002

For further information see the YouTube video by Scott Page.

1.1 *The diversity prediction theorem*

The diversity prediction theorem states that, under the conditions stated above, the squared error in the answer of the crowd equals the average individual squared error minus the variance in individual answers.

To do the assignments in this section, you will need to be able to perform calculations with summations of sequences. This is explained in section 16.1.

1. Example: guessing the amount of chips
   In the local supermarket, grocery shoppers can win 'a lifetime of chocolate' if they correctly guess the amount of chocolate chips in a bowl.
   At the last moment, Mariam decides to participate in the competition. She looks at the answer sheet which shows the answers of all previous participants. So far, the guesses have been 350, 111, 263, 345, 600, 555, 431, 98, 777, 582 and 201 chips. Mariam is reminded of the 'wisdom of the crowd' and decides to write down the average of these entries as her own answer.

   a. Calculate Mariam's answer.
      After Mariam writes down her answer, the supermarket manager closes the competition. She announces that the correct answer was 387 chips. Mariam has won a lifetime of chocolate!
   b. Why was Mariam's strategy effective in this case?
   c. Do the conditions of the diversity prediction theorem apply in this case?

2. Example: betting at the FIFA World cup
At the FIFA World Cup, spectators can guess in which minute the first goal will be made. If their guesses are above 90 minutes, this means they think the first goal will be made in the second match. Their guesses are: 88, 12, 131, 25, 69, 37, 51, 99, 73, 5, 72, 30, 11, 29, 8, 13, 25, 26 and 31 minutes.
It turns out that the first goal is made in the 46^{th} minute.

a. Discuss whether the conditions for the diversity prediction theorem hold in this case.

b. We denote the answer of the ith individual by x_i. What is the value of x_3?

c. The crowd's answer is the average of the individual answers, formally denoted by:

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{1.1}$$

where n is the group size and x_i are the individual answers. Calculate this value.

d. The squared error of the crowd's answer is the squared difference between the average, $\overline{x}$ and the true value, t:

$$(\overline{x} - t)^2 \tag{1.2}$$

Calculate this value.

e. An individual's squared error is the squared difference between the individual answer and the true answer: $(x_i - t)^2$. What is the squared error of individual 3?

f. The average individual squared error is the average of the squared differences between the individual answers, x_i and the true answer, t:

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - t)^2 \tag{1.3}$$

What is the value of this quantity?

g. The variance in individual answers is given by the squared distance to their mean answer:

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 \tag{1.4}$$

Calculate this quantity.

h. Combining the expressions in (1.2) to (1.4) gives a mathematical expression for the diversity prediction theorem:

$$(\overline{x} - t)^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - t)^2 - \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 \tag{1.5}$$

Show that this relationship indeed holds for the considered example (use three decimals accuracy).

3. Exploring the theorem

   a. Why should we consider average *squared* errors rather than average errors?
   b. Show that (1.5) is true when all individuals give the same answer.
   c. Argue that, when not all individuals give the same answer, (1.5) implies that the error in the group answer (the expression in (1.2)) is smaller than the expected error in an individual answers (the expression in (1.3)).
   d. Suppose that all individual errors, $x_i - t$, are large, is it possible to still have an accurate crowd answer? Explain.
   e. Suppose that the group consists of individuals that are all experts. Since they all base their answer on the same background knowledge, the variance in their answers is small. Is it possible that the error in their average answer is large? Explain.
   f. Is it possible for a group of laymen to be more accurate than a group of experts? Why, or why not?

4. Proof of the theorem

   a. To prove the theorem, we start by writing the first expression on the right hand side of (1.5) in a different way. Show that:

$$(x_i - t)^2 = (x_i - \bar{x})^2 + (\bar{x} - t)^2 + 2 \cdot (x_i - \bar{x}) \cdot (\bar{x} - t) \qquad (1.6)$$

   b. Show that

$$\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x}) \cdot (\bar{x} - t) = 0 \qquad (1.7)$$

   c. Use the results of the previous subquestions to prove the theorem.

5. For each of the following situations, discuss if they satisfy the conditions of the Diversity Prediction Theorem.

   (a) Average ratings of films and books on websites like IMDb and Goodreads.
   (b) At a county fair, 787 people guessed the weight of an ox to win a prize. They could write their guess on a card and give it to the organisers.
   (c) In order to find a sunken ship or crashed plane, the US Office of Naval Intelligence asks experts from various disciplines to give their ideas about where the ship or plane might be. These experts may for instance be climate scientists, physicists, oceanographers and engineers. The Office could then for instance average the possible locations given, or look for commonalities in the answers.
   (d) The university organises a panel discussion with 25 students and teachers to predict the outcome of the national elections.

2
Cultural evolution

Culture is developed by learning and transmission between generations. Although culture is not limited to humans, the ability of humans to shape and accumulate cultural information over generations is unique. Scientists believe that the transmission of cultural information has strongly affected human evolution. Thus, whereas evolutionary changes in most biological species are likely to be preliminary determined by genetics and mechanisms such as mutation and natural selection, social learning, and the interaction between genetics and culture, is believed to be an important factor in the evolutionary history of humans[1]. Over the past decades there has been an enormous increase in studies of cultural evolution, in biology, economy, and anthropology. There is a growing awareness that not only do genetic components play a role in cultural inheritance, but also that cultural transmission may affect our genetic constitution. A famous example is the prevalence of lactose intolerance in certain geographical regions, in relation to the tradition of milk consumption: whereas only about 4% of the Swedish are lactose intolerant, about 90% of the Chinese are. It is believed that the traditional pastoral culture in northern Europe has exerted selection pressures on this genetic trait. It is believed that gene-culture coevolution has had an important effect on human social behaviour and the way human societies are shaped[2]. Mathematical models are used to investigate how such gene-culture interactions work, and to what extend they can explain observed patterns of contemporary human behaviour.

See Stanford Encyclopedia for a review of this subject.

[1] P.J. Richerson and R. Boyd. *Not by Genes Alone - How culture transformed human evolution*. The University of Chicago Press, Chicago and London, 2005; R. Boyd and P.J. Richerson. *The Origin and Evolution of Cultures*. Oxford University Press, 2005; and F.J. Odling-Smee K.N. Laland M.W. Feldman. *Niche Construction*. Monographs in Population Biology. Princeton University Press, 2003

See: global lactose intolerance distribution, and this article in Plus Magazine, on Natural selection, Maths and Milk.

[2] H. Gintis. Gene-culture coevolution and the nature of human sociality. *Phil. Trans. R. Soc. B*, 366:878–888, 2011

2.1 Cultural accumulation

Historical studies of human culture reveal that there are periods of exponentially increasing cultural output, such as poems, musical compositions, or scientific discoveries. The question is what causes this. Can it be explained by an increased effectiveness of transmission of cultural traits, through social learning, or is it an effect of an increased creativity, or both? Is it due to only behavioural changes, or is it likely to involve genetical changes, that are favoured by natural selection? Or is it due to a combination of behaviour and genetics?

Enquist et al[3] studied these possible explanations for the exponential growth of cultural traits by means of a mathematical model, that we will consider in this section. In their model the amount of culture that a human society contains at a given time is represented by a variable, x. It represents the total number of cultural traits that are shared by a substantial amount of individuals in a population. This is inevitably a simplification, but it allowed them to study the separate and interactive effects of social learning and genetics on the accumulation of cultural traits over successive generations.

[3] M. Enquist S. Ghirlanda A. Jarrick C.A. Wachtmeister. Why does human culture increase exponentially? *Theoretical Population Biology*, 74:46–55, 2008

Cultural transmission between generations occurs through demonstrations and teaching, but also through social learning. A necessary condition for adequate transmission is the ability to remember transmitted culture accurately. There is, inevitably, a loss of culture in the course of time. For instance, not all books, poems, stories, or paintings that were ever produced still exist today. This is also true for scientific elements, such as, for instance, mathematical proofs, or ways to prepare certain medicines. To account for this, their model contains a loss rate, λ. The production of new cultural traits per time unit is given by another quantity, γ, which represents creativity. The dynamics of x are determined by these two quantities.

Using this model, they first examined the separate effects of loss rate and creativity, to see if these mechanisms by themselves could explain the observed exponential growth of cultural traits. Then they added genetical evolution, assuming that, through mutation and selection, the efficiency of social transmission and creativity increases over generations, and studied whether this could explain the observations. Finally, they examined the dynamics of cultural traits with a gene-culture co-evolution model, where it is assumed that the way in which creativity is affected by existing cultural traits is an evolving, genetically determined trait.

2.1.1 *A discrete time version of the model*

To do the assignments in this section, study the background theory of sections 9.1, 9.2, and chapter 10.

The original model of Enquist et al. is formulated in continuous time. In this section we consider a discrete time version, where time steps represent successive generations.

1. The basic model
   The initial model looks as follows:
   $$x(n+1) = x(n) - \lambda \cdot x(n) + \gamma \tag{2.1}$$
   a. What are the possible ranges of values for the two parameters, given their interpretation?
   b. As mentioned, one time step represents one generation. $x(n)$ is the number of cultural traits in the nth generation. What are the units of the parameters λ and γ, expressed in generations and number of cultural traits?
2. A situation without creativity

a. Give the model equation for a situation where there is no creativity.

b. What is the equilibrium value of x for this situation?

c. Is this a stable or an unstable equilibrium?

The loss rate is assumed to be positive. You might examine this numerically, trying out several initial conditions.

d. What is the solution equation for $x(n)$ in this case?

e. What does a graph of $x(n)$ versus n look like in this situation?

f. Can there be cultural accumulation without creativity according to this model?

3. A situation without cultural transmission
In this situation, it is assumed that γ is positive, but there is no cultural transmission.

   a. What is the value of λ in this case?

   b. What does the model look like in this case?

   c. What happens in this situation?

   d. Can there be cultural accumulation without cultural transmission?

4. Combined effects of transmission and creativity
We now consider the situation where creativity is positive, and at least some culture is transmitted to the next generation.

   a. What is the equilibrium value $\hat{x}$ in this case?

   b. Is the equilibrium stable?

   c. What does the model predict that will happen with $x(n)$ in the long run?

   d. Sketch a graph of $x(n)$ for a value of $x(0) < \hat{x}$.

   e. Can this model explain the observed phenomenon of cultural accumulation?

   f. What is the solution equation for this model?

5. Adding genetic evolution of cultural transmission
The previous results have shown that there must be additional mechanisms that affect cultural accumulation. One possibility is that the efficiency of cultural transmission may change due to genetic changes. For instance, the ability for social learning, or the ability to transfer knowledge effectively may evolve. If evolution is efficient, the end result would be perfect transmission. We study the model's predictions concerning cultural accumulation in that final situation.

   a. What is the value of λ in this situation?

   b. What is the recurrence equation for $x(n)$ in this case?

   c. What is the solution equation?

   d. On the basis of this result, would you say that a model with evolving efficiency of cultural transmission can explain exponential increase in x, or not? Motivate your answer.

6. Genetic evolution of creativity
   If creativity has positive fitness effects, the frequency of genes that enhance this trait will increase in the course of time. We will assume that, due to such genetical changes, the creativity in a population will increase at a constant rate c:

$$\gamma(n+1) = \gamma(n) + c \tag{2.2}$$

   a. What is the solution equation for $\gamma(n)$?
   b. Substitute this solution in the recurrence equation for $x(n)$ in (2.1).
   c. Examine numerically, using Excel, R, or another device, what happens with $x(n)$ in the course of time. Choose your own values of λ, γ, and c. What happens according to this model when n becomes very large?
   d. Can the observed cultural accumulation be explained by genetic evolution of creativity?

7. Cultural evolution of creativity
   Creativity may also be enhanced by cultural, rather than genetic changes, since culture may improve cognitive and/or practical skills. It has been shown, for instance, that human brains change in response to training and education [4]. In addition, creativity benefits from innovations such as literacy, mathematical and logical thinking techniques, and computer technology. This may be incorporated in the model by assuming a relationship between creativity and the value of $x(n)$, for instance:

$$\gamma\,(n) = \gamma + b \cdot x(n) \tag{2.3}$$

   with $b > 0$.

   a. Substitute this expression for γ in the recurrence equation for $x(n)$ in (2.1).
   b. What is the equilibrium value of x?
   c. For which parameter combinations is this a positive value?
   d. Show that the equilibrium stable if it is positive.
   e. Summarise in a concise way how, according to these results, transmission and cultural evolution of creativity affect the amount of cultural traits.
   f. What happens if the equilibrium value of x is negative?
   g. What is the shape of $x(n)$ in this case?
   h. What does this mean in terms of amount of cultural traits, transmission and cultural evolution of creativity?
   i. Can this form of cultural evolution explain the observed accumulation of culture?

[4] M. Tomasello. *The Cultural Origins of Human Cognition*. Harvard University Press, London, 1999

We will assume that $\lambda \neq b$.

8. Gene-culture coevolution of creativity
   The previous analysis shows that the observed accumulation of culture can be explained by cultural evolution of creativity. In order for this to work, creativity needs to respond to cultural traits.

This means that culture-dependent creativity should be favoured by natural selection. To examine under which conditions this happens, we consider a model where b in (2.3) evolves due to genetic changes:

$$\begin{aligned} \gamma(n) &= \gamma + b(n) \cdot x(n), \text{ with} \\ b(n+1) &= b(n) + \kappa \text{ and} \\ x(n+1) &= x(n) - (\lambda - b(n)) \cdot x(n) + \gamma \end{aligned} \tag{2.4}$$

where $\kappa > 0$.

a. Since genetic evolution depends on the rate at which mutations occur, and on natural selection, it is much slower than the rate at which behavioural processes, such as cultural transmission and invention take place. This means that, to approximate the shape of the function $x(n)$, we can first solve the recurrence equation for $x(n)$ under the assumption that $b(n)$ is constant. What is the solution equation for $x(n)$ if $b(n) = b$?

b. What is the solution equation for $b(n)$?

c. If $b(n)$ is substituted in the solution equation for $x(n)$, we find an expression for the cultural accumulation in time when there is gene-culture coevolution of creativity. Make a graph of this function (use Excel, R or another device) for $\lambda = 0.05, \gamma = 1, \kappa = 0.0001$, with $b(0)$ and $x(0)$ equal to zero, for n up to 300 generations. Indicate the value of $\frac{\gamma}{\lambda}$ in the plot.

d. Discuss the evolution of $b(n)$, in relation to the loss rate λ, and the implications for the evolution of cultural accumulation: are there any major changes in cultural accumulation? If so, when do these occur? How does this relate to the evolution of creativity, and the loss rate?

2.1.2 *The continuous time model*

In this section we consider the original, continuous time version of Enquist et al.'s model.

To do the assignments in this section, study the background theory of sections 9.1, 9.2, and 9.2.1, and chapter 12.

1. The basic model
   The initial model looks as follows:

$$\frac{dx}{dt} = -\lambda \cdot x + \gamma \tag{2.5}$$

   a. What are the possible ranges of values for the two parameters, given their interpretation?

   b. What are the units of the parameters if time is expressed in years?

2. A situation without creativity

   a. What does the model look like when there is no creativity?

b. What is the equilibrium value of x for this situation?

c. What type of equilibrium is this?

d. What does a graph of $x(t)$ look like in this situation?

e. Can there be cultural accumulation without creativity according to this model?

f. Solve the differential equation for this model, with starting value $x(0) = x_0$.

3. A situation without cultural transmission
If the time unit is the average expected lifetime of individuals, and the loss rate is $\lambda = 1$, there is no cultural transmission.

a. What does the model look like in this case?

b. What is the equilibrium value?

c. What type of equilibrium is it?

d. What does the model predict will happen in this case?

e. What does a graph of $x(t)$ look like in this case? Consider different possible starting conditions.

f. Can there be cultural accumulation without cultural transmission?

g. Solve the differential equation for this model, with starting value $x(0) = x_0$.

4. Combined effects of transmission and creativity.
We now consider the situation with positive creativity, and $\lambda < 1$.

a. What does the model predict that will happen with $x(t)$ in this case?

b. Sketch a graph of $x(t)$ for a value of $x(0) < \hat{x}$.

c. Can this model explain the observed phenomenon of cultural accumulation?

d. Solve the differential equation for this model, with starting value $x(0) = x_0$.

5. Adding genetic evolution of cultural transmission
The previous results have shown that there must be additional mechanisms that affect cultural accumulation. One possibility is that the efficiency of cultural transmission may change due to genetic changes. For instance, the ability for social learning, or the ability to transfer knowledge effectively may evolve. If evolution is efficient, the end result would be perfect transmission. We study the model's predictions concerning cultural accumulation in that final situation.

a. What is the value of λ in that situation?

b. What is the differential equation for x in this case?

c. Show that the value of $x(t)$ increases linearly with time in this case.

d. On the basis of this result, would you say that a model with evolving efficiency of cultural transmission can explain exponential increase in x, or not? Motivate your answer.

6. Genetic evolution of creativity
If creativity has positive fitness effects, the frequency of genes that enhance this trait will increase in the course of time. It can be shown that, due to such genetical changes, the creativity in a population may increase at a constant rate c:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = c \tag{2.6}$$

a. What type of function is $\gamma(t)$?

b. What is the differential equation for x in this case?

c. If we denote the starting value $\gamma(0)$ by γ, the solution of the differential equation is:

$$x\left(t\right) = \frac{a}{\lambda} + \left(x_0 - \frac{a}{\lambda}\right) \cdot e^{-\lambda t} + \frac{c}{\lambda} \cdot t,$$
$$\text{with } a = \gamma_0 - c/\lambda \tag{2.7}$$

d. What happens to the expression with the exponential when t becomes very large? What does this apply for the approximate value of $x(t)$ at large t-values?

e. Can genetic evolution of creativity account for cultural accumulation?

7. Cultural evolution of creativity
Creativity may be enhanced by cultural, rather than genetic changes, since culture may improve cognitive and/or practical skills. It has been shown, for instance, that human brains change in response to training and education. [5] In addition, creativity benefits from innovations such as literacy, mathematical and logical thinking techniques, and computer technology. This may be incorporated in the model by assuming a relationship between creativity and the value of x, for instance:

[5] M. Tomasello. *The Cultural Origins of Human Cognition*. Harvard University Press, London, 1999

$$\gamma\left(x\right) = \gamma + b \cdot x \tag{2.8}$$

with $b > 0$.

We will assume that $\lambda \neq b$.

a. What is the differential equation for x in this case?

b. What is the equilibrium value of x?

c. For which parameter combinations is this a positive value? Is the equilibrium stable in this case?

d. What does this mean in terms of amount of cultural traits, transmission and cultural evolution of creativity?

e. What happens if the equilibrium value of x is negative?

f. What is the shape of $x(t)$ in this case?

g. What does this mean in terms of amount of cultural traits, transmission and cultural evolution of creativity?

h. Can this form of cultural evolution explain the observed accumulation of culture?

8. Gene-culture coevolution of creativity
The previous analysis shows that the observed accumulation of culture can be explained by cultural evolution of creativity. In order for this to work, creativity needs to respond to cultural traits. This means that culture-dependent creativity should be favoured by natural selection. To examine under which conditions this happens, we consider a model where b in (2.8) evolves:

$$\begin{aligned} \gamma(x) &= \gamma + b \cdot x, \text{ so:} \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= -(\lambda - b) \cdot x + \gamma, \text{ with} \\ \frac{\mathrm{d}b}{\mathrm{d}t} &= \kappa \end{aligned} \tag{2.9}$$

with $\kappa > 0$.

a. Since genetic evolution depends on mutation rate, and natural selection, it is much slower than the rate at which cultural transmission and invention takes place. This means that, to approximate the shape of the function $x(t)$, we can first solve the differential equation for x under the assumption that b is constant. Show that the solution is:

$$x(t) = \frac{\gamma}{\lambda - b} + \left(x_0 - \frac{\gamma}{\lambda - b}\right) \cdot e^{-(\lambda - b)t} \tag{2.10}$$

b. Find the expression for $b(t)$ by solving the first differential equation in (2.9), with $b(0) = b_0$.

c. If $b(t)$ is substituted in (2.10), we find an expression for the cultural accumulation in time when there is gene-culture coevolution of creativity. Make a graph of this function for $\lambda = 0.05, \gamma = 1, \kappa = 0.0001$, with b_0 and $x(0)$ equal to zero, for t up to 400 time units. Indicate the value of $\frac{\gamma}{\lambda}$ in the plot.

d. Discuss the evolution of b, in relation to the loss rate λ, and the implications for the evolution of cultural accumulation.

3
Arms races

Written in collaboration with Jurre Honkoop

Arms races are periods of competition for superiority in quality and quantity of weapons. These periods can occur both in times of peace or during war. Arms races are associated with negative consequences ranging from economic inefficiency to war. Money spent on arms races could be spent more efficiently if countries had some basis on which they could trust one another.

See also: Why do mathematicians play games?, and The cold war arms race.

Arms races tend to lead to a negative spiral that is hard to break as is illustrated by most notably the cold war. The Soviet Union and the United States tried to outcompete each other's weapon production, a competition that arguably resulted from grievances between communism and capitalism and fear of each other's arsenal that was developed during the second world war. Another example of such a negative spiral can be found in the Pre-first World War naval arms race between the UK and Germany. The British were very concerned with Germany's fast naval growth and decided to 'retaliate' by increasing naval potential themselves. The arms race arguably contributed to the first world war, if not by starting it by increasing its gravity and destructive consequences.

As arms races tend to have such frightening and burdensome consequences it is essential to understand the phenomenon in order to be able to counter it. Models could help in finding what caused the arms race in question and what different interventions would change in the arms race dynamics if anything. One of the first to model the phenomenon mathematically was Richardson. He was a meteorologist, who devoted himself to the ideals of peace while teaching at a British university from 1940 until his death in 1953. He developed models to examine how budgets, threats, and grievances between countries may create conditions for arms races to occur. His work was published posthumously[1]. The Richardson arms race model is one of the most influential models in its field, providing a basis for many more complex (game theoretic) models.

[1] L.F. Richardson. *Arms and Insecurity: A Mathematical Study of the Causes and Origins of War*. Boxwood Press, Pittsburgh, 1960

3.1 A discrete time version of the Richardson arms race model

The background theory for the assignments in this section is provided in sections 9.1, 9.2, and chapters 10 and 11.

Richardson's original model was formulated in continuous time, and consists of a system of differential equations that describe the dynamics of the amount of weaponry of different countries. We will here study a discrete time analogue of the model.

1. A model for two countries
   Let $x(n)$ denote the total armament of country 1 at time n, and $y(n)$ that of country 2. It is assumed that the countries react to each other's strength by adjusting their weaponry proportionally:

$$\begin{aligned} x(n+1) &= x(n) + a_{12} \cdot y(n) \\ y(n+1) &= y(n) + a_{21} \cdot x(n) \end{aligned} \tag{3.1}$$

   a. What range of values can the state variables x and y assume?
   b. What range of values may the parameters a_{12} and a_{21} be assumed to have in this context?
   c. Explain what a value of a_{12} or a_{21} that is larger than one would imply in terms of a country's reaction to others.
   d. In the rest of this assignment we assume that these parameters are smaller than one. The system of difference equations is a two-dimensional linear system. What is its equilibrium value?
   e. Write the system of equations in (3.1) in matrix form.
   f. Is the equilibrium stable or unstable?
   g. Is the equilibrium a spiral?
   h. What does this model predict that will happen when at least one of the initial values $x(0)$ or $y(0)$ is positive?

a_{12} and a_{21} are assumed to be smaller than one.

2. Adding grievances to the model
   Nations that are warlike or hold old grievances against other nations might build a certain amount of arms anyway, even if there is no threat by others, this is modelled by adding a constant amount of arms per time step, for each of the countries:

$$\begin{aligned} x(n+1) &= x(n) + a_{12} \cdot y(n) + g_1 \\ y(n+1) &= y(n) + a_{21} \cdot x(n) + g_2 \end{aligned} \tag{3.2}$$

   a. What values can the grievances assume?
   b. Does this model have an (interpretable) equilibrium?
   c. Do the state variables move towards the point that you have calculated in the previous subquestion, or away from it?
   d. What are the predictions concerning the arms race if $x(0)$ and $y(0)$ are both zero?

3. Adding fatigue to the model
   Finally, it is assumed that arms are lost due to budget deficits or budget costs:

$$\begin{aligned} x(n+1) &= x(n) - c_1 \cdot x(n) + a_{12} \cdot y(n) + g_1 \\ y(n+1) &= y(n) - c_2 \cdot y(n) + a_{21} \cdot x(n) + g_2 \end{aligned} \tag{3.3}$$

This model may be written as

$$\begin{aligned} x(n+1) &= f_1 \cdot x(n) + a_{12} \cdot y(n) + g_1 \\ y(n+1) &= f_2 \cdot y(n) + a_{21} \cdot x(n) + g_2 \end{aligned} \tag{3.4}$$

The parameters f_1 and f_2 are called *fatigue coefficients*.

a. Explain why the fatigue coefficients should lie in the interval $[0, 1]$.

b. What does a value of a fatigue coefficient equal to zero mean, and what does it mean if it is equal to one?

c. Write the system of difference equations in (3.4) in matrix form.

d. What are the equilibrium equations for this model?

e. After World War 1 Richardson collected data for, among others, Britain (country 1) and Italy. The data are as follows: $f_1 = 0.25, a_{12} = 0.3, g_1 = 0.05, f_2 = 0.75, a_{21} = 0.2, g_2 = 0.1$. What is the equilibrium value in this situation?

f. Is this a stable equilibrium?

g. What is the prediction concerning an arms race between these countries?

h. The collected data for Germany (country 1) and Italy are: $f_1 = 1, a_{12} = 0, g_1 = 0.15, f_2 = 0.75, a_{21} = 0, g_2 = 0.1$ Write down the system of recurrence equations for this situation.

i. What do you predict on the basis of this model?

4. A model for three countries
In many situations more than one country are involved in arms races. Extending the model to three countries gives:

$$\begin{aligned} x(n+1) &= f_1 \cdot x(n) + a_{12} \cdot y(n) + a_{13} \cdot z(n) + g_1 \\ y(n+1) &= f_2 \cdot y(n) + a_{21} \cdot x(n) + a_{31} \cdot z(n) + g_2 \\ z(n+1) &= f_3 \cdot z(n) + a_{31} \cdot x(n) + a_{32} \cdot y(n) + g_3 \end{aligned} \tag{3.5}$$

a. Write this system of equations in matrix form.

b. The coefficients of the matrix, that were measured by Richardson for Germany (country 1), England (country 2) and Italy are as follows:

$$\begin{pmatrix} 1 & 0.1 & 0 \\ 0.2 & 0.25 & 0.3 \\ 0 & 0.2 & 0.75 \end{pmatrix} \tag{3.6}$$

What does this say about the relationship between these three countries at that time?

c. What does it say about the budget that these countries spend on the maintenance of their weaponry?

d. The measured grievances are $g_1 = 0.15, g_2 = 0.05, g_3 = 0.1$. Is there an equilibrium?

e. Examine numerically whether there is an arms race in this situation if initially none of these countries have any weapons.

5. Richardson's data.
After World War I, Richardson collected data on ten nations. He came up with the following matrix. The diagonal represents the coefficients f_i, and the off-diaginal elements correspond to a_{ij}.

	Czech	China	France	Germany	England	Italy	Japan	Poland	USA	USSR
Cz	1/2	0	0	1/10	0	0	0	1/20	0	0
Ch	0	1/20	0	0	0	0	1/5	0	0	1/10
Fr	0	0	0	1/10	1/5	0	1/5	0	0	0
Ger	1/5	0	1/5	1	1/10	0	0	1/20	0	1/25
Eng	0	0	0	1/5	1/4	3/10	1/10	0	0	0
It	0	0	1/10	0	1/5	3/4	0	0	0	1/10
Ja	0	1/5	0	0	0	0	1/2	0	1/5	1/5
Po	1/20	0	0	1/20	0	0	0	1/2	0	1/20
USA	0	0	0	1/10	1/10	1/10	1/5	0	13/20	1/10
USSR	0	1/10	0	2/5	1/10	1/10	1/5	1/20	0	1/2

The values that he measured for the grievances are (in the same order of the nations as given in the table):

$$(1/20, 1/20, 1/20, 3/20, 1/20, 1/10, 3/20, 1/20, 1/20, 1/10)$$

Examine arms races between pairs of countries of your own choice, by means of computer simulations. Interpret the outcome of the simulations by means analysing the stability of equilibria of the model.

These data are given by William W. Farr on `http://www.math.wpi.edu/Course_Materials/MA2071A99/Projects/arms/project_template.html`.

4
Economic growth

Written in collaboration with Victoria Smit

The study of the aggregate of economic systems, *macroeconomics*, is concerned with the forecasting of national income on the basis of major economic factors, such as, for instance employment, national income and expenditure, or inflation level, and political measures such as taxes and investment. Economists use intricate mathematical models that predict the dynamics of GDP under different circumstances. Policy decisions are based, among other things, on such model predictions.

The state of the economy for a geographic area is usually quantified by means of its *Gross Domestic Product* (GDP). This is a measurement of the productive activity for that geographic area (usually a country), defined over a specific time interval. There are three main definitions of GDP:

1. The sum of final sales within the region and time period. For instance, the total sum of final sales of goods and services sold to consumers and firms.
2. The sum of value added within the region and time period. Value added is the product that a firm created by transforming materials and unfinished goods.
3. The sum of factor incomes earned from economic activities within the region and time period. With this definition, rather than looking at the goods and sales, the incomes are considered a measurement of economic output.

To avoid double counting, only final sales are included.

Using the approach 1, based on expenditures, the value of GDP, denoted by Y becomes:

$$Y = C + I + G + (X - Z) \tag{4.1}$$

where C denotes the household consumption expenditure, I business investment in goods and additions to inventory stock, G government expenditures on salaries and goods, X gross exports, and Z gross imports.

The GDP of countries tends to grow in the long run. On shorter time scales there are significant and recurring fluctuations around the GDP trend, referred to as business cycles. Business cycles are characterised by peaks (after expansions) and troughs (after recessions). Most European Countries have an average annual growth rate of GDP per capita ranging between 1.4 and 1.9%[1].

[1] M. Burda and C. Wyplosz. *Macroeconomics: A European Text*. Oxford University Press, 2013

Neoclassic economic growth theory to a large extent uses aggregate optimisation models, that are based on the concept of *production functions*. These models describe the relationship between the aggregate product output (measured e.g. by GDP) and productive inputs such as labour, capital, knowledge, etc. The models consist of one or more differential or difference equations that describe the dynamics of the quantities in (4.1). Optimal economic growth strategies are derived from these models by means of optimisation methods[2]. In static optimisation models, such as the Solow-Swan model[3] considered in the next section, the dynamics of business cycles are ignored.

[2] N. Hritonenko and Y. Yatsenko. *Mathematical Modeling in Economics, Ecology and the Environment*. Springer Optimization and Its Applications. Springer-Verlag Berlin Heidelberg, 2013

[3] R. Solow. A contribution to the theory of economic growth. *Quarterly Journal of Economics*, 70:65–94, 1956

4.1 *The Solow-Swan model*

This model is one of the most celebrated models in economic growth theory. It is the foundation for many, more complex, neoclassical growth models. The model assumes that economic output, Y is distributed between household consumption, and investment in capital:

$$Y = C + I \tag{4.2}$$

The background theory for the assignments in this section is provided in sections 9.1, 9.2, chapter 12, and section 19.3.

The output Y is generated by a production function, that depends on the amount of labour, L and capital stock K:

$$Y = F(K, L) \tag{4.3}$$

An interactive demonstration of this model can be found on the Wolfram demonstrations website: SolowGrowthModel.

Figure 4.1 shows a flow diagram of the model.

The model considers dynamics of the variables Y, I, C, K, L in continuous time, assuming that capital, K, changes according to the following differential equation:

$$\frac{\mathrm{d}K}{\mathrm{d}t} = I - \delta \cdot K, \text{ with } \delta > 0 \tag{4.4}$$

Government and trade expenditures are ignored in this model. Note that all the variables in this equation depend on time.

The parameter δ is called the *depreciation rate* of the capital. This is due to the fact that part of the capital will need to be used to replace for instance old capital goods, such as buildings, machinery, vehicles, or equipment.

The part, s of the output that is invested is called the *saving rate*:

$$s = \frac{I}{Y} \tag{4.5}$$

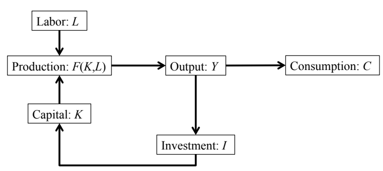


Figure 4.1: Flow diagram of the Solow-Swan model

1. The production function
   The form of $F(K, L)$ remains to be specified. The Solow-Swan model is quite general: it can be used with different possible shapes of production functions, provided that they have a certain general shape. As an example, we will consider the so-called two-factor *Cobb-Douglas production function*, which has the following form:

$$F(K, L) = A \cdot K^{\alpha} \cdot L^{1-\alpha}, \text{ with } 0 < \alpha < 1,\ A > 0 \tag{4.6}$$

The parameter A is the total factor productivity, which reflects the level of technology. α is called the *elasticity* of output with respect to capital. Given that α is smaller than 1, there will be diminishing marginal productivity. This means that the more capital is used, the smaller the increase in output will eventually be.

a. What is the percentage increase in productivity according to this production function, if capital increases by 1% and $\alpha = 0.9$?

b. The function $F(K, L)$ in (4.6) may also be written as:

$$F(K, L) = L \cdot f(k), \text{ with } k = \frac{K}{L} \tag{4.7}$$

Derive an expression for $f(k)$. Note that $k \geq 0$. The Solow-Swan model assumes that $F(K, L)$ has the form given in (4.7), and that $f(k)$ satisfies certain conditions. We will examine these for the production function in (4.6).

c. The first condition is that the function $f(k)$ increases in k. Show that this is indeed true.

d. The second condition is that the derivative of the function in the point 0 should be ∞. Show that this is indeed so.

e. The third condition is that the function should be concave. Show that this is indeed so.

f. Sketch a graph showing the general shape of the function $f(k)$.

2. A model with constant labour
The full Solow-Swan model includes a differential equation for labour, L, but we will first consider a situation where L is assumed to be constant in time.

a. Recall that $Y = F(K, L)$. Derive an expression for the relationship between I and $f(k)$, using equations (4.5) and (4.7).

b. Show that the dynamics of the capital-labour ratio $k = K/L$ are described by:

Use the rules for differentiation in section 19.1.

$$\frac{\mathrm{d}k}{\mathrm{d}t} = s \cdot f(k) - \delta \cdot k \tag{4.8}$$

3. The steady state
A steady state is a situation where either all variables are constant in time, or they all change at the same rate.

a. Show that the variables Y, C, I, K are all constant when:

$$s \cdot f(k) = \delta \cdot k \tag{4.9}$$

b. Sketch $s \cdot f(k)$ and $\delta \cdot k$ in one graph, and argue that (4.9) has two solutions. Which one is the relevant steady state value?

c. Show that the value $k = 0$ is an unstable equilibrium value of k.

See section 12.3.

d. The steady-state value of k is a stable equilibrium, since the value of $\frac{dk}{dt} > 0$ for k-values that lie to the left hand side of this point, and $\frac{dk}{dt} < 0$ for values that lie to the right hand side of this point. Argue on the basis of the graph that you made in 3b that this is so.

4. The golden rule
Combining the knowledge of the steady state and GDP, one could start asking questions about what the ideal value of the saving rate s would be. To answer this question, we should return to Eq. (4.2). We know that consumption affects the economic well-being. As an experiment, try to think of a situation where you save all your income. You spend nothing on consumption. While the interest rate on your bank account will actually enlarge your personal capital, you may not be well-off. After all, you need to eat, pay rent, etc. The amount of money that is available for this is the consumption C. Optimal economic growth is attained at the value of s that maximises the per capita consumption:

Recall that the saving rate is the proportion of the output, Y, that is allocated to investment, I.

$$c = \frac{C}{L} \tag{4.10}$$

a. We denote the steady state values by $\hat{I}, \hat{k}$, etc. Derive from (4.4) that:

$$\hat{I} = \delta \cdot \hat{K} \tag{4.11}$$

b. Use this result and (4.2) to show that:

$$\hat{c} = f(\hat{k}) - \delta \cdot \hat{k} \tag{4.12}$$

c. Derive an equation for the optimal value of $\hat{k}$. Denote this value by k^*. The equation is called the *golden rule of capital accumulation*.

d. Use the previous results and (4.9) to derive the corresponding *golden-rule saving rate*:

$$s^* = k^* \frac{f'(k*)}{f(k^*)} \tag{4.13}$$

e. What is the value of s^* for the Cobb-Douglas production function in (4.6)?

f. Use (4.9) to calculate k^* for this production function.

5. A model with labour growth at a constant rate
Up to now we have considered situations where the amount of labour, L, remains constant. In reality, this value will change, subject to population growth, and age structure. The complete Solow-Swan model also includes a differential equation for the dynamics of labour, namely:

$$\frac{dL}{dt} = \eta \cdot L, \text{ with } \eta \geq 0 \tag{4.14}$$

This assumes a constant population growth rate. The situation where $\eta = 0$ was considered above. We will now assume that $\eta > 0$.

a. Solve the differential equation, and derive an expression for $L(t)$, with a starting value of $L(0)$.

See section 12.2.

b. From the product rule for differentiation, it follows that:

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{1}{L}\frac{\mathrm{d}K}{\mathrm{d}t} - \frac{K}{L^2}\frac{\mathrm{d}L}{\mathrm{d}t} \tag{4.15}$$

Use this to derive the following differential equation for k:

This is called the *fundamental equation of the Solow-Swan model.*

$$\frac{\mathrm{d}k}{\mathrm{d}t} = s \cdot f(k) - (\delta + \eta) \cdot k \tag{4.16}$$

c. Derive an equation for the steady-state value of k. As before, we will denote this value by $\hat{k}$.

d. Argue that there are two solutions, one at $k = 0$ and one at the steady-state value, and show that $k = 0$ is an unstable equilibrium. (See question 1f.)

e. Argue that the process will converge to the steady-state value, i.e. that it is a stable situation.

f. In the previously considered situation, where $\eta = 0$, the steady-state value $\hat{k}$ corresponded to a situation where all of the variables are constant in time. In the current situation, L grows at the rate $\eta > 0$. Show that in the steady state all variables, K, Y, I, C grow at the same rate.

g. In the steady state situation, K grows at a constant rate η, so when $k = \hat{k}$:

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \eta \cdot K \tag{4.17}$$

Use (4.4), (4.2), and (4.7) to show that:

$$\hat{c} = f\left(\hat{k}\right) - (\delta + \eta) \cdot \hat{k} \tag{4.18}$$

where $\hat{c}$ is the per capita consumption at the steady state.

h. Show that the optimal capital-labour ratio, k^* satisfies:

This is the form of the golden rule of capital accumulation with $\eta > 0$, see 4c.

$$f'(k^*) = \delta + \eta \tag{4.19}$$

i. Show that the golden rule saving rate is as stated in (4.13).

j. Show that for the Cobb-Douglas production function defined in (4.6) we have:

$$s^* = \alpha, \text{ and } k^* = \left(\frac{A \cdot s^*}{\delta + \eta}\right)^{\frac{1}{1-\alpha}} \tag{4.20}$$

6. Properties of neoclassical production functions
The Solow-Swan model is an example of a *neoclassical* economic model. In this type of models it is assumed that the production function $F(K, L)$ is neoclassical. This means that it should have certain properties, which guarantee that the steady state value $\hat{k}$ exists, and is positive and stable. These properties are:

I Essentiality of inputs:

$$F(K, 0) = F(0, L) = 0 \tag{4.21}$$

II Positive returns: $F(K, L)$ should increase in K and in L, implying that both partial derivatives are positive:

$$\frac{\partial F}{\partial K} > 0, \frac{\partial F}{\partial L} > 0 \tag{4.22}$$

See section 19.3 for an explanation of partial derivatives.

III Diminishing returns: the function should be concave in both variables, implying that the second partial derivatives should be negative:

$$\frac{\partial^2 F}{\partial K^2} < 0, \frac{\partial^2 F}{\partial L^2} < 0 \tag{4.23}$$

IV Constant returns to scale:

This is called *linear homogeneity* of the function F.

$$F\left(x \cdot K, x \cdot L\right) = x \cdot F\left(K, L\right), \text{ for any } x > 0 \tag{4.24}$$

V The *Inada conditions* should hold:

$$\begin{aligned} \lim_{K \to 0} \frac{\partial F}{\partial K} &= \infty, \lim_{L \to 0} \frac{\partial F}{\partial L} = \infty, \\ \lim_{K \to \infty} \frac{\partial F}{\partial K} &= 0, \lim_{L \to \infty} \frac{\partial F}{\partial L} = 0 \end{aligned} \tag{4.25}$$

Consider the two-factor Cobb-Douglas production function in (4.6).

7. Show that this function satisfies the first condition (see 6I) stated above.
8. Show that this function has positive returns (6II).
9. Show that this function has diminishing returns (see 6III).
10. Show that this function is linearly homogeneous (see 6IV).
11. Show that the Inada conditions are satisfied for this function (see 6V).

5
Climate

See also Plus magazine articles: Climate modelling, and Climate change.
More informantion on climate modelling can be found on the websites of M. E, Brunke and UCL.

Originally, climate sciences were concerned with the study of the earth's climatic history. This was done with geological, geographical, and botanical methods. At a later stage, in the 1950-ies, measurements of weak radioactivity of various isotopes allowed for the dating of organic material, and determination of flux rates in environmental systems. The study of (chemical and biological) tracers in ice cores from Antarctica and Greenland produced an accurate reconstruction of the historical chemical composition of the Earth's atmosphere. Additional information on climatical history was provided by studying ocean and lake sediments, tree rings, mineral deposits in caves.

As the amount and detail of quantitative data on climatic processes increased, questions arose concerning the underlying physical, chemical, and biological mechanisms. This created the need for the development of quantitative, mechanistic climate models. The goal of such models is to test hypotheses about past climatic developments and their causes, but also the magnitude and impact of future climate change. The models consider one or more of the major components of the Earth's climate system:

1. *Atmosphere*: This is the gaseous part above the earth's surface, containing gaseous, and traces of liquid, and solid substances. Processes that occur in the atmosphere are for instance the formation of clouds, precipitation, transport of heat. The upper atmosphere is called the *photosphere*. This part is especially important with respect to global temperature.

2. *Hydrosphere*: All forms of water above or below the earth's surface. Processes include water flow, but also heat transport, and exchange of gases with the atmosphere. The ocean is the most important reservoir of carbon with fast turn-over rates.

3. *Cryosphere*: All forms of ice in the system. Processes in the cryosphere have an important effect on radiation balance (due to reflection of solar radiation), and ocean salinity. This in turn affects flows and temperature in the hydrosphere.

4. *Land Surface*: The solid part of the earth. Processes in this part are for instance the movement of continents, transformation and

reflection of radiation, transfer of momentum and energy.

5. *Biosphere*: Organic cover of land masses (vegetation and soil) and aquatic organisms. Processes in the biosphere are for instance exchange of carbon between different reservoirs, vegetation-effects on humidity, radiation reflection, and momentum exchange between atmosphere and land surface.

The human-determined processes, which may be called *anthroposphere*, are usually considered separately from the Earth's climate system.

Climate models differ in which of these components are included, and in the amount of detail concerning the processes that are modelled. The development of models is a strongly interdisciplinary science, comprising domains of physics, chemistry, geography, and biology. The prediction of future climate development, and the impact of the anthroposphere are issues of imminent importance, that are being examined with complex computer models. The relatively simple models, such as we will consider here, however, do illustrate some of the main mechanisms involved in climate change. Furthermore, they will give you an impression of the way in which climate models are constructed and analysed.

The website of Brian Rose, at the University of Albany contains examples of more complex climate models. The American Chemical Society website has a Climate Science Toolkit with information on energy balance, planetary temperatures, and models for earth and other planets.

5.1 Energy Balance Model

To do the assignments in this section you need the background knowledge of chapters 9 and 12.

Global energy balance models consider the effect of the balance between incoming and outgoing radiation on the earth's average surface temperature.

1. Model formulation and parameters
   The earth's atmosphere is modelled as a thin spherical air layer, with a thickness of h. We denote the earth's radius by R.

   a. The volume of a sphere with a radius r equals $\frac{4}{3}\pi \cdot r^3$. Compute the volume of the atmospherical layer.

   b. Since the earth's radius R is much larger than the thickness of the atmospheric layer h, we can simplify the result, and approximate the volume, by ignoring terms with h^3 or h^2. Show that the approximate volume equals: $4\pi \cdot R^2 \cdot h$.

   The (average) density of air in the atmosphere is denoted by ρ, and the *specific heat* of air by c. If the globally averaged surface temperature equals T, the thermal energy content of the atmosphere equals:

   $$4\pi \cdot R^2 \cdot h \cdot \rho \cdot c \cdot T \tag{5.1}$$

   The incoming solar energy, consisting of short-wave radiation, is assumed to reach the earth through a circular disc, with a surface of $\pi \cdot R^2$. A proportion $1 - \alpha$ of the incoming solar radiation is reflected back, as short-wave radiation, into space.

The proportion α is called the earth's *albedo*. If we denote the solar radiation per time unit by S_0, the total amount of solar energy that reaches the surface per time unit is:

$$\pi \cdot R^2 \cdot S_0 \cdot (1 - \alpha) \tag{5.2}$$

Albedo varies in time and location. For instance, ice, or desert landscapes reflects more solar radiation than forests. For now, we will consider the average albedo, and model it as a constant.

A part of the energy is sent back into space as *long-wave back radiation*. The outgoing thermal energy depends on the surface temperature, T, the earth's surface area, $4\pi \cdot R^2$ and the so-called 'planetary emissivity', ϵ, and equals:

$$4\pi \cdot R^2 \cdot \epsilon \cdot \sigma \cdot T^4 \tag{5.3}$$

where σ denotes the so-called *Stefan-Boltzmann constant* of physics.

The resulting model for the change in the earth's thermal energy per time unit is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(4\pi \cdot R^2 \cdot h \cdot \rho \cdot c \cdot T\right) = \pi \cdot R^2 \cdot S_0 \cdot (1 - \alpha) - 4\pi \cdot R^2 \cdot \epsilon \cdot \sigma \cdot T^4 \tag{5.4}$$

c. Assuming that only T changes in time, and all the other quantities are constant, derive the following differential equation for T:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{h \cdot \rho \cdot c}\left(\frac{S_0 \cdot (1 - \alpha)}{4} - \epsilon \cdot \sigma \cdot T^4\right) \tag{5.5}$$

d. As stated, albedo, α is the proportion of solar radiation that is reflected back into space. What does this tell you about its possible range of values, and its dimension?

e. Look up the value and unit of σ on the internet.

f. The quantity S_0 is called the *solar constant*. It is the average solar radiation that the earth receives per time unit and per surface area. Its approximate value is 1367 $\mathrm{W/m^2}$. The emissivity, ϵ is a dimensionless quantity, with value approximately equal to 0.6.

g. What should be the unit of T to make the expression between the brackets in (5.5) consistent?

h. What are the units of the expression between brackets in (5.5), and what is its dimension?

i. What should be the dimension of the product $h \cdot \rho \cdot c$ to make the model equation (5.5) dimensionally consistent?

j. The atmospheric layer that is considered has a thickness of 8.3 km, the (average) air density ρ equals 1.2 $\mathrm{kg/m^3}$, and the specific heat of air c is about 1000 J/ (kg K). Calculate the value of $h \cdot \rho \cdot c$, expressed in the proper units.

k. What is the unit of time if we use these values for the quantities involved?

l. With what value should we multiply the right-hand side of the differential equation if we want to express time in days, rather than seconds?

2. Earth's predicted surface temperature
   In this assignment you may use the numerical parameter values that were given in the previous exercise.

   $h = 8.3$ km, $\rho = 1.2$ kg/m^3, $c = 1000$ J/ (kg K)

   a. What is the equilibrium surface temperature of the earth $\hat{T}$ as a function of albedo, according to this model, in degrees K (use two decimal precision)?
   b. Is this a stable or an unstable equilibrium?
   c. Sketch a graph of $\hat{T}$ versus $1 - \alpha$ for the whole range of possible values of α.
   d. What would be the earth's predicted temperature in degrees Celsius if none of the solar radiation is reflected, and all is absorbed?
   e. What would be the earth's predicted temperature in degrees Celsius if all of the solar radiation is reflected?
   f. At which value of the albedo α would the predicted temperature be zero degrees C?
   g. The earth's average albedo is estimated (by means of remote sensing) to be about 0.3, and its average temperature is 14°C. Based on this, would you say that the model is accurate?
   h. A possible reason for discrepancies is inaccuracy of the estimate of emissivity ϵ. What value of ϵ would agree with an average temperature of 14°C? State the answer with four decimals accuracy.
   i. Fill in the following table. Use the estimate for ϵ that you have derived in the last assignment.

parameter	meaning	range or value	units	dimension
h				
ρ				
c				
ϵ				
σ				
S_0				
α				

3. Long-wave back radiation feedback
   The effect of changes in conditions, such as atmospheric CO_2 depends on a complex interaction of different processes that affect the radiation balance. The strength of feedback quantifies the contribution of a single process as a response to a disturbance of the balance. A large part of climate modelling research is aimed at measuring feedback parameters for different processes. We will apply this method in this assignment, to examine the effects of the earth's long-wave back-radiation according to the model of (5.4). The usual notation for the model in (5.5) is as follows:

$$C \cdot \frac{\mathrm{d}T}{\mathrm{d}t} = \frac{S_0 \cdot (1 - \alpha)}{4} - \epsilon \cdot \sigma \cdot T^4 \tag{5.6}$$

where the constant C corresponds to $h \cdot \rho \cdot c$. To be consistent with climate modelling literature, will use this same notation here.

Note that this does not influence expressions for equilibria. Also, since $C > 0$, conclusions concerning the stability of equilibria that are based on the expression on the right-hand side remain valid.

a. We will denote the equilibrium for this model by $\hat{T}_0$. Give the general (i.e. non-numerical) expression for $\hat{T}_0$.

b. We will call the expression on the right hand side of (5.6) $f(T)$. Show that the equilibrium $\hat{T}_0$ is stable, by means of $f'(T)$.

   The value of $f'\left(\hat{T}_0\right)$ is called the *Planck feedback*. It is generated by the reaction of the earth's surface to incoming solar radiation, by means of long-wave outgoing radiation. This can be considered as a *feedback loop* . The fact that it is negative shows that it has a cooling down effect.

c. How would the differential equation look if this feedback was missing?

d. What would happen to the earth's surface temperature?

e. What is the value of the Planck feedback for this model, using the parameter values in your table and an albedo of $\alpha = 0.3$?

f. What are the units and dimension of the Planck feedback?

The equilibrium value of the surface temperature was given in 2h.

   The value of the feedback provides information on how the earth's surface temperature would react to a disturbance, caused by, for instance, an increase in atmospheric CO_2. Suppose, for instance, that there is a small, but permanent, change in conditions, that add an amount Q to the surface temperature. As a consequence, the equilibrium temperature will shift too. Due to the feedback caused by back radiation, however, the temperature will not increase by Q, but by a smaller amount.

   To distinguish the earth's surface temperature after the change in conditions from that before, we will denote it by T_b. This temperature is determined by the differential equation:

$$C \cdot \frac{\mathrm{d}T_b}{\mathrm{d}t} = f\left(T_b\right) + Q \tag{5.7}$$

   where $f()$ is the same function as before (see right hand side of (5.6)). If we denote the new equilibrium by $\hat{T}_b$, the equilibrium equation is:

$$f\left(\hat{T}_b\right) + Q = 0 \tag{5.8}$$

   A first-order Taylor expansion may be used to approximate $f(T_b)$ in the vicinity of the point $\hat{T}_0$:

See section 19.4.

$$f\left(\hat{T}_b\right) \approx f\left(\hat{T}_0\right) + f'\left(\hat{T}_0\right) \cdot \left(\hat{T}_b - \hat{T}_0\right) \tag{5.9}$$

g. Explain why $f\left(\hat{T}_0\right) = 0$, and use this fact to simplify this equation.

h. Fill in the approximation in (5.8) and determine the approximate value of $\hat{T}_b$.

i. What is the predicted effect of an addition in temperature of $Q = 2°C$ on the surface temperature?

Note that a change of $1°C$ is the same as a change of $1K$.

4. Cirrus cloud feedback
Beside the long-wave back radiation there are many more processes involved in the earth's surface temperature regulation. For instance, the atmosphere is more accurately modelled by a multi-layered system, that each have their own albedo, and emissivity of long-wave radiation. Each of these processes has its own feedback component, that affects the reaction to external conditions. As an example we consider a two-layered system, by adding a layer with cirrus clouds to the model. It is assumed that these do not affect short-wave radiation reflection, so the surface albedo stays the same, but they do affect long-wave radiation. Let β represent the fraction of the layer covered by cirrus clouds. This means that a fraction β of the long-wave emission from earth hits the cirrus clouds. We model the cirrus clouds as a *black body*, assuming that it has an emissivity of 1, and assume that long-wave radiation is equal in all directions. This means that half of the emitted radiation goes back to earth, and half goes into space. As a result, the earth's surface temperature model becomes:

$$C \cdot \frac{\mathrm{d}T}{\mathrm{d}t} = \frac{S_0 \cdot (1-\alpha)}{4} - \varepsilon \cdot \sigma \cdot T^4 + \frac{1}{2}\beta \left(\varepsilon \cdot \sigma \cdot T^4\right) \tag{5.10}$$

The last expression in this equation acounts for the fact that a proportion $\frac{\beta}{2}$ of the long wave radiation that is released by the earth's surface comes back through emission from the cirrus clouds.

a. Give an expression for the equilibrium value of this model, $\hat{T}_1$.
b. Calculate the value of $\hat{T}_1$ if the cloud cover is 10%. Give the result with two decimal accuracy, in Kelvin and $°C$.

The outcome is clearly not so realistic, which shows the model's shortcomings with respect to the effect of clouds. There are all sorts of complications, such as effects of clouds on albedo, different forms of clouds, and feedback of temperature on cloud formation. Also, the atmosphere has more than just two layers, as was assumed in this model.

You should find that the equilibrium temperature in this model is considerably higher than that of the previous one. Obviously, the cirrus-cloud layer diminishes the effect of the negative feedback by the long-wave back radiation from the surface. To distinguish the contributions of these two different processes, two different feedback components are considered: the Planck feedback, and the cloud feedback.

First note that the differential equation in (5.10) can be written as:

$$C \cdot \frac{\mathrm{d}T}{\mathrm{d}t} = f(T) + f_c(T) \tag{5.11}$$

where $f(T)$ is the original model function of (5.6) and $f_c(T)$ is the additional component due to the cirrus cloud layer. Accordingly, the derivative to T of the right-hand side is the sum of derivatives $f'(T) + f_c'(T)$. These different derivatives, evaluated in the equilibrium $\hat{T}_1$ correspond to respectively the Planck feedback for this model, and the cloud feedback.

c. Calculate the values of these two feedback parameters, using the value for $\hat{T}_1$ that you have found in 4b.

d. If we call the Planck feedback λ_p and the cloud feedback λ_c, the approximate effect of a change in conditions of magnitude Q degrees would be to shift the equilibrium to:

$$\hat{T}_{1,b} \approx \hat{T}_1 - \frac{Q}{\lambda_p + \lambda_c} \tag{5.12}$$

This can be shown in exactly the same way as in 3h.

What would be the predicted effect of a change with $Q = 2$ degrees?

5. Albedo feedback

So far we have assumed that albedo α has a constant value. Reflection of short-wave radiation by the earth's surface is, however, largely determined by how much of the surface is covered by ice and vegetation. Since this, in turn, is affected by the surface temperature, it creates another feedback loop. Models such as these are based on the idea of daisyworld[1]. Daisyworld is a world filled with two different types of daisies: black daisies and white daisies. They differ in albedo, which is how much energy they absorb as heat from sunlight. White daisies have a high surface albedo and thus reflect light and heat, thus cooling the area around them. See Wood et al.[2] for a review. For an example of a two-variable model see Rombouts[3]. The model may be extended, to include the reaction of vegetation on temperature, and its effects on albedo. To examine the effect of albedo, the basic model of (5.6) is adjusted, by making α a function of T:

[1] A.J. Watson and J.E. Lovelock. Biological homeostasis of the global environment: the parable of daisyworld. *Tellus B*, 35(4):286–289, 1983

[2] A.J.Wood G.J. Ackland J.G. Dyke H.T.P. Williams T.M. Lenton. Daisyworld: a review. *Reviews of Geophysics*, 46:1–23, 2008

[3] J. Rombouts and M. Ghil. Oscillations in a simple climate-vegetation model. *Nonlinear Processes in Geophysics*, 22: 275–288, 2015

An interactive demonstration of the model is given on the website DaisyBall.

$$C \cdot \frac{\mathrm{d}T}{\mathrm{d}t} = \frac{S_0 \cdot (1 - \alpha(T))}{4} - \epsilon \cdot \sigma \cdot T^4 \tag{5.13}$$

There are several ways to define the function $\alpha(T)$. Here we will use a simple piece-wise linear function:

$$\alpha(T) = \begin{cases} 0.8 \text{ if } T < 220 \text{ K} \\ a + b \cdot T \text{ if } 220 \text{ K } \leq T \leq \text{ 300 K} \\ 0.2 \text{ if } T > 300 \text{ K} \end{cases} \tag{5.14}$$

where a and b are constants, chosen in such a way that the function is continuous. The value of $\alpha = 0.8$ represents the maximum albedo of the earth's surface, when all the oceans are frozen. The value $\alpha = 0.2$ is the minimum albedo when there is no ice, and no vegetation.

a. Calculate the values of a and b such that at $\alpha(220) = 0.8$ and $\alpha(300) = 0.2$, and sketch a graph of $\alpha(T)$ versus T.

b. The equilibrium equation for this model is:

$$\frac{S_0 \cdot (1 - \alpha(\hat{T}))}{4} - \varepsilon \cdot \sigma \cdot \hat{T}^4 = 0 \tag{5.15}$$

There is one equilibrium, $\hat{T}_1$ at a value smaller than 220, what is its value?

c. Briefly describe what would be the state of the earth in this equilibrium. The model in (5.13) may be written as:

$$C\frac{dT}{dt} = f_a(T) + f_p(T) \tag{5.16}$$

where $f_a(T)$ denotes the contribution of albedo to the change in T, and $f_p(T)$ that of the long-wave back radiation. The albedo feedback corresponds to the derivative of $f_a(T)$ and the Planck feedback to that of $f_p(T)$.

Note that in the model of (5.6) $f_a(T)$ is a constant, so the definition of the Planck feedback is the same.

d. What are the values of respectively the albedo feedback, and the Planck feedback in this equilibrium?
e. Is this a stable or an unstable equilibrium? Motivate your answer.
f. Show that the values $\hat{T}_2 \approx 235.71$ and $\hat{T}_3 \approx 288.69$ are also equilibria of this model.
g. What are the values of respectively the albedo feedback, and the Planck feedback in $\hat{T}_2$?
h. What does it imply concerning the effects of the different temperature regulation processes?
i. The fact that the albedo feedback parameter is positive, means that a change in temperature (in either direction) is enhanced by changes in albedo. Why is this so?
j. What type of equilibrium is this?
k. Compute the feedbacks in the equilibrium $\hat{T}_3$. What can you concluding the stability of this equilibrium?
l. What happens according to this model, given that the current average earth surface temperature is about 14°C?
m. What does the model predict would happen if, by additional processes, the earth's temperature reaches 27°C?

Fill in this value of T in the differential equation. What is the direction of change?

n. What are the values of the feedback parameters at a temperature of 301°K? What does it imply concerning the effects of the different temperature regulation processes?

6 Bioaccumulation

Written in collaboration with Eveline van Woensel

When organic chemicals such as pesticides, or tiny particles such as micro-plastics are released into the environment they will be taken up by organisms. This uptake may occur through several pathways. Firstly, there is respiratory uptake, through lungs or gills. Chemicals can also be introduced into the body through skin exposure and/or the ingestion of food in the form of other already contaminated animals. Chemicals are eliminated from the body through respiration, excretion and metabolism, causing biotransformation. Bioaccumulation occurs when organisms take up chemicals at a higher rate than they eliminate them. Persistent toxic chemicals (PBTs) and persistent organic pollutants (POPs) have an especially high potential for bioaccumulation, since their low water solubility and high lipid solubility makes elimination difficult. Even when environmental concentrations of the toxicant are relatively low, bioaccumulation can lead to chronic poisoning, obstructing reproduction, growth and survival. The potency of POPs to harm the environment was discovered in the 1960s, and attention to solving this issue has since then increased. Methods of estimating the risk of bioaccumulation are now an important part of environmental risk assessments of factories with potential chemical discharge. Environmental risk assessment is largely based on mathematical models such as those considered in this chapter[1].

See Effects of bioaccumulation on ecosystems and Ecotox models.

[1] N.P.E. Vermeulen R. van der Oost, J. Beyer. Fish bioaccumulation and biomarkers in environmental risk assessment: a review. *Environmental Toxicology and Pharmacology*, 13:57–149, 2003

Models of bioaccumulation are used to predict the impacts of chemicals that are released into the environment, either for individual organisms or for whole ecosystems and at different trophic levels. Over the years, several models have been constructed to predict the concentration of chemicals in organisms in relation to the amount released. Toxicokinetic models capture information about the uptake, distribution and elimination of a chemical in individual organisms. They are used to predict toxic effects in organisms, but are also components of more complex models for predicting bioaccumulation in foodwebs.

6.1 A toxicokinetic model for fish

The background theory for the assignments in this section is found in chapters 9 and 12.

There are several types of toxicokinetic models. So-called 'single compartment models' consider the individual as a single, well-mixed vessel. Thus, they either assume that the chemical is evenly distributed over the organism, or only consider a part of the organism as being relevant with respect to toxicity. More complex models consider variation of concentrations in different body parts. In this section, we consider a single-compartment model for the concentration of a toxic substance in fish, which was developed by Gobas[2]. This model is used in environmental risk assessment for aquatic toxicants.

[2] F.A.P.C. Gobas. A model for predicting the bioaccumulation of hydrophobic organic chemicals in aquatic food-webs: application to lake ontario. *Ecological Modelling*, 69:1–17, 1993

1. Initial model: effects of respiration
   We denote the concentration of the contaminant in the fish by C_f and the dissolved contaminant concentration in the water by C_{wd}. As mentioned above, exchange between the organism and its environment occurs through several pathways. In first instance, consider a relatively simple model, where exchange only occurs through respiration, let k_1 be the rate of uptake through the gills, and k_2 the elimination rate through the same pathway. In that case there is a constant inflow and outflow of toxicants, and the change in C_f may be described by the following differential equation:
   $$\frac{\mathrm{d}C_f}{\mathrm{d}t} = k_1 \cdot C_{wd} - k_2 \cdot C_f \tag{6.1}$$
   a. If the concentration C_f is given in μg per kg fish, time is given in days, and C_{wd} in μg per litre, what should be the units of respectively k_1 and k_2?
   b. Which ranges of values can the parameters $C_{wd}, k_1,$ and k_2 assume?
   c. Assuming that the concentration in the water, C_{wd} is constant, what is the equilibrium concentration of the toxicant in the organism, $\hat{C}_f$?

   In this context, the equilibrium is usually called the *steady state*.

   d. Is this a stable or an unstable equilibrium? Motivate your answer.
   e. For which values of k_1 and k_2 will the concentration of the toxicant in the organism be larger than that in the surrounding water?
2. Including effects of excretion, metabolism and growth
   Elimination of the toxicant occurs through several other mechanisms as well: through egestion of faecal matter, with rate k_e, and metabolic processes, with rate k_m.
   a. Adjust the model accordingly.
   b. What is the equilibrium value of C_f according to this model? Is it a stable equilibrium?
   c. For which parameter combinations is the steady state value of C_f larger than the environmental concentration?

   The concentration C_f is also affected by growth of the organism, since it depends on the volume of the organism. To examine this, let V denote the volume of a fish, and assume that it grows at a constant rate k_g, i.e.:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = k_g \cdot V \tag{6.2}$$

If we let x denote the amount of ingested toxicant, the concentration within the organism is $C_f = \frac{x}{V}$.

d. Assuming that x is constant, show that the change in C_f due to growth is given by:

$$\frac{\mathrm{d}C_f}{\mathrm{d}t} = -k_g \cdot C_f \tag{6.3}$$

Note that $\frac{\mathrm{d}V}{\mathrm{d}t}$ is the same as $V'(t)$ and apply the chain rule. See section 19.1.

Including the growth term in the differential equation gives:

$$\frac{\mathrm{d}C_f}{\mathrm{d}t} = k_1 \cdot C_{wd} - \left(k_2 + k_e + k_m + k_g\right) \cdot C_f \tag{6.4}$$

e. What is the steady state value of C_f?

f. Describe what happens, according to this model if a fish is transferred from clean to polluted water with a constant value of C_{wd}.

g. What are the dimensions of $k_2, k_e, k_m,$ and k_g, and what range of values can these parameters assume?

3. Including food uptake
According to Gobas' model, there is an additional input source of the toxic substance, namely ingestion through food. This is assumed to occur at a constant rate k_d.

a. Assume that the concentration of the toxicant in food sources is constant, at a value of C_{food}. Adjust the model to include this.

b. The parameter C_{food} is given in $\frac{\mu\mathrm{g}}{\mathrm{kg}}$. What are the units of k_d?

c. Calculate the steady state value $\hat{C}_f$ and show that it is a stable equilibrium.

d. The expression $\frac{k_1}{k_2+k_e+k_m+k_g}$ is called the *bioconcentration factor* and $\frac{k_d}{k_2+k_e+k_m+k_g}$ is called the *biomagnification factor*. How do these factors relate to the steady state value of C_f?

4. The bioaccumulation factor BAF
Since a part of the toxicant is ingested through food intake by all organisms, the concentrations accumulate throughout the food chain. The bioaccumulation factor is an important environmental risk measure that quantifies this process[3].

The *bioaccumulation factor* is defined by:

$$\mathrm{BAF} = \frac{\hat{C}_f}{C_{wd}} \tag{6.5}$$

[3] J.A. Arnot and F.A.O.C. Gobas. A review of bioconcentration factor (bcf) and bioaccumulation factor (baf) assessments for organic chemicals in aquatic organisms. *Environmental Reviews*, 14:257–297, 2006

a. What are the units of BAF?

b. Find an expression for the BAF based on the steady state expression for $\hat{C}_f$ that was computed in 3c.

5. Bioavailability

A large part of empirical research in biological risk assessment is devoted to quantification of toxicological models such as these in different circumstances. The values of the parameters depend on characteristics of the toxicant, temperature, physiological characteristics of the fish species, and ecosystem characteristics such as food web structure. Gobas made parameter estimations for the lake Ontario system. We will consider his approach as an example here. Not all the pollutant substance that is present in aquatic systems may be taken up by organisms. The interaction of chemicals with organic matter may result in complexes that are too large to permeate membranes such as present in gills of fish. The concentration C_{wd} corresponds to the truly dissolved concentration of the chemical in the water. This is a fraction of the total chemical concentration C_{wt}:

$$C_{wd} = BSF \cdot C_{wt} \tag{6.6}$$

The factor BSF is called the *bioavailability* of a toxicant.

a. What values can BSF assume?

b. What is the dimension of BSF?

The value of BSF depends on an important chemical characteristic, called the *1-octanol-water partition coefficient*, which is usually denoted by K_{ow}. Its value depends on the specific substance and the temperature. Values for K_{ow} for specific substances, measured at a certain temperature may be found in the literature, or on websites. They are calculated from the outcomes of specific chemical experiments. Gobas considered a K_{ow} value of $10^{6.6}$, for PCBs at a mean water temperature of 8°C. He derived the following expression for the bioavailability:

$$BSF = \left(1 + \frac{K_{ow} \cdot [OM]}{d_{om}}\right)^{-1} \tag{6.7}$$

$[OM]$ denotes the concentration of organic matter in the water: the weight of organic matter in kg, per litre of water. d_{om} is the density of organic matter, i.e. its weight in kg, per litre of matter.

c. Show that K_{ow} is a dimensionless quantity.

d. The estimates values for lake Ontario are $[OM] = 2.5 \cdot 10^{-7}$kg/l and $d_{om} = 0.9$kg/l. Give an estimated value of BSF for the considered PCBs, with an accuracy of four decimals.

e. What is the value of C_{wd} in $\frac{\mu g}{L}$,if the total concentration of the chemical in the water is $1.1 \cdot 10^{-9}$g/L?

6. Gill uptake rate
The rate at which a chemical is absorbed through the gills depends of the gill ventilation rate, G_v, the uptake efficiency E_w and the fish' (wet) weight, V_f:

$$k_1 = \frac{E_w \cdot G_v}{V_f} \tag{6.8}$$

   a. If the units of G_v are L/day, and V_f is given in kg, what are the units of E_w?

   E_w appears to be related to K_{ow}. The value of K_{ow} varies for different substances and temperatures. By varying this value, and measuring the corresponding value of E_w, researchers have derived the following relationship:

$$\frac{1}{E_w} = \frac{G_v}{Q_w} + \frac{G_v}{Q_l} \cdot \frac{1}{K_{ow}} \tag{6.9}$$

   where Q_w is the transport of the chemical in aqueous components of the fish, and Q_l the transport in lipid substances. Both are rates, with units L/day.

   b. Show that this equation is dimensionally consistent.
   c. Use (6.8) to derive an expression for the relationship between k_1 and K_{ow}.
   d. Gobas argues that in most siutations Q_l is approximately $Q_w/100$, and Q_w (in L/day) is related to the weight of a fish (in kg) as follows:

$$Q_w \approx 88.3 \cdot {V_f}^{0.6} \tag{6.10}$$

   Use this information to estimate the value of k_1 for a fish of 200 g, at the considered K_{ow} value of $10^{6.6}$. Give your final answer with an accuracy of 1 decimal, and give the units.

Note that this equation expresses an empirically fitted curve, and not a functional relationship: it is not dimensionally correct. We consider it here because it is often used in practice.

7. Food uptake
(See Arnot and Gobas[4]) To calculate food uptake, the following relation may be used:

$$k_d = \frac{E_d \cdot F_d}{V_f} \tag{6.11}$$

where E_d is the dietary chemical transfer efficiency, F_d the feeding rate and V_f the wet weight.

   a. The parameter F_d has units kg/day. What are the units of E_d?
   b. Estimate the value of E_D from K_{ow} as follows:

$$E_d = \left(3 \cdot 10^{-7} \cdot K_{ow} + 2\right)^{-1} \tag{6.12}$$

   c. The feeding rate (in kg/day) is related to temperature (°C) and weight (kg) as follows:

$$F_d \approx 0.022 \cdot {V_f}^{0.85} \cdot e^{0.06 \cdot T} \tag{6.13}$$

   Calculate the feeding rate at a temperature of 8°C for a fish of 200 g,

[4] J.A. Arnot and F.A.P.C. Gobas. A food web bioaccumulation model for organic chemicals in aquatic ecosystems. *Environmental Toxicology and Chemistry*, 23:2343–2355, 2004

Recall that the units of k_d were determined in 3b.

Note that this equation expresses an empirically fitted curve, and not a functional relationship: it is not dimensionally correct. We consider it here because it is often used in practice.

d. Use (6.11) to find an estimate for k_d for a fish of 200 g at a water temperature of 8°C. Give your answer with 3 decimals accuracy and give the proper units.

8. Elimination rates
The gill uptake rate k_1 and its elimination rate k_2 are proportional. Gobas gives the following relationship for this:

$$\frac{k_1}{k_2} = L_f \cdot K_{ow} \tag{6.14}$$

where L_f is the ratio of the lipid weight of the fish in kg, divided by the wet weight in kg.

a. Show that this equation is incorrect with respect to units.

To make the equation consistent with respect to units, the right hand side should be multiplied by the weight of water volume per kg:

$$\frac{k_1}{k_2} = L_f \cdot K_{ow} \cdot W \tag{6.15}$$

b. What is the value of W and what are its units?

c. This extra parameter is often not mentioned explicitly in the literature, but it is important to include it, to prevent mistakes in calculations when different units are used. To illustrate this, suppose that k_1 would be given in dL/kg, what should then be the proper value and units of W?

d. Estimate k_2 for a fish with 10% lipid content.

The value of k_1 was estimated in 6d.

e. The faecal elimination rate, k_e is approximately a quarter of the food uptake rate k_d. There are as yet not many data available on metabolic transformation rate k_m, for organic substances in fish. For the current model, we will assume that it equals about 0.0006/day. The elimination due to growth may, at a temperature of 8°C, be approximated by:

The value of k_d was estimated in 7d.

$$k_g \approx 0.000502 \cdot V_f{}^{-0.2} \tag{6.16}$$

Give an approximation for the sum $k_e + k_m + k_g$, with an accuracy of three decimals

9. Estimation of the BAF

a. Estimate the bioaccumulation factor on the basis of your estimates, using the relationship you derived in 4. Use a value of $5.7 \cdot 10^{-6} \frac{\mu g}{kg}$ for C_{food}

b. The 10-log-values of BAF quantities measured in the field are in the range of about 2 to 8. Does your outcome fall in this range?

In practice, more elaborate procedures are needed, taking into account food-web structure and diet composition

6.2 Models for PCB bioaccumulation in mussels

In the previous section, fish were treated as a one-compartment system. More realistic models take into account that organisms

The background theory for the assignments in this section is found in chapters 9, 12, and 13.

consist of multiple tissues and organs that have different properties with respect to uptake and excretion of toxic substances. Such models consider an organism as a multi-compartment system, with possibly differing concentrations of the toxicant. The models considered in this section were developed by Yu and coworkers[5] on the basis of detailed experiments. They examined the PCB bioaccumulation in mussels empirically, and fitted several different types of models. One of these was a two-compartment model.

[5] K.N. Yu P.K.S. Lam C.C.C. Cheung C.W.Y. Yip. Mathematical modeling of pcb bioaccumulation in perna viridis. *Marine Pollution Bulletin*, 45: 332–338, 2002

1. A single compartment model
   The first model that was examined considered the mussel's soft tissue as a single compartment. The concentration of PCB in the tissue is denoted by Q. The model equation is:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = R \cdot W - k \cdot Q \tag{6.17}$$

   R denotes the uptake rate, W de concentration of PCB in the water, measured in ppb (parts per billion), and k is the elimination rate. Time was measured in days, Q was measured in μg g^{-1} PCB per lipid mussel tissue.

   ppb is a unit often used in ecotoxicology, it corresponds to μg/L.

   a. What ranges of values can the parameters R, W, and k assume?
   b. What are the units of k and R?
   c. How many μg of PCB does a litre of water contain if $W = 5$ ppb?
   d. What is the equilibrium value of Q?
   e. Is this a stable or an unstable equilibrium?
   f. Denote the initial value of Q by Q_0. Show that the following function is a solution $Q(t)$ of the differential equation in (6.17):

$$Q\left(t\right) = \frac{RW}{k} + e^{-kt}\left(Q_0 - \frac{RW}{k}\right) \tag{6.18}$$

   Show that the derivative satisfies the differential equation.

   g. Figure 6.1 shows the typical shape of the measured PCB concentration in mussel soft tissue as a function of time since the start of the experiment. Does this model fit the data? Motivate your answer.

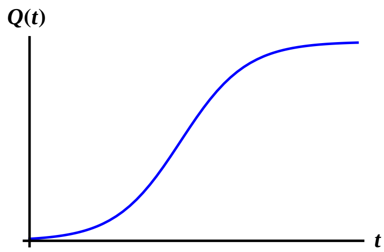


Figure 6.1: General shape of the measured $Q(t)$ function.

2. A model with two compartments
   The next model that was considered made a distinction between the concentrations in two separate compartment Q and P. In the first two-compartment model it is assumed that the toxicant is taken up in compartment one, and may be transported from there to compartment two, but not vice versa, whereas excretion occurs at different rates from each of the compartments:

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} &= R \cdot W - (k_1 + \alpha)\, Q \\ \frac{\mathrm{d}P}{\mathrm{d}t} &= \alpha \cdot Q - k_2 \cdot P\end{aligned} \tag{6.19}$$

   a. What are the units of the parameters α, k_1, and k_2?

b. What ranges of values can they assume?

c. What is the equilibrium value?

d. Is this equilibrium stable or not?

e. Note that the first differential equation in (6.19) does not depend on P. Show that its solution equals:

$$Q(t) = \frac{RW}{k_1 + \alpha} + e^{-(k_1+\alpha)t}\left(Q_0 - \frac{RW}{k_1 + \alpha}\right) \tag{6.20}$$

f. Can this model explain the data? Motivate your answer.

3. The third model that was examined assumes that there are two compartments with exchange, but that excretion only occurs from the first:

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}t} &= R \cdot W - (k + \alpha)\, Q + \beta \cdot P \\ \frac{\mathrm{d}P}{\mathrm{d}t} &= \alpha \cdot Q - \beta \cdot P \end{aligned} \tag{6.21}$$

a. What are the units of the parameters $\alpha, k,$ and β?

b. What ranges of values can they assume?

c. What is the equilibrium value?

d. Is this equilibrium stable or not?

e. What type of model is (6.21)?

f. Discuss the fit with the observed data pattern, given what you know about the possible dynamics of this type of model.

4. The final model that they tried was a one compartment model with a time-varying uptake rate:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = R \cdot W - k \cdot Q \tag{6.22}$$

where the uptake rate R was assumed to be related to Q, and thus change in time:

$$R = \frac{\gamma \cdot Q}{Q + \delta} \tag{6.23}$$

a. Write up the complete model equation, by substituting the expression for R.

b. What are the equilibria $\hat{Q}_1$ and $\hat{Q}_2$?

c. For what parameter combinations is the second equilibrium positive?

d. Examine the stability of both equilibria, under the condition that the second equilibrium is positive.

e. Based on this result, do you think that this model will fit the data pattern?

7
Epidemics

Written in collaboration with Kevin de Wit

Infectious diseases are transmitted by contact between infected and susceptible individuals. The agents that cause such a disease may be micro-organisms, such as viruses, or bacteria, or multicellular organisms, such as parasites. For many diseases the discovery of the agent that causes the disease and the types of contact that lead to transmission were important milestones in human history. They opened the way to devising strategies that prevent the outbreak of major epidemics.

See: Math Plus article: Mathematics of infectious diseases, The Center for Disease Control and Prevention: Statistics on infectious diseases, Nrich: Interactive model simulations, University of Cambridge: School project.

John Snow, who discovered in 1854 that Cholera was transmitted through drinking water, is considered to be the founder of modern epidemiology. He was able to end an outbreak of the disease by disinfecting a water pump with chlorine. Depending on the disease, its transmission, and the state-of-the art in medicine, modern-day prevention methods consist of disinfection, quarantine, and vaccination. Important questions in this context concern the dynamics of the disease, for example: Is an outbreak imminent, or already on its way? Has the disease been successfully eradicated? Other practical questions address the most effective way of fighting a disease: How many should be vaccinated? How effective would imposing a quarantine be?

Mathematical methods were introduced in epidemiology at the beginning of the 20th century. Kermack and McKendrick[1] published the first mechanistic model for the spread of infections diseases. This model provides the basis for many, more complex models that are being used today (for a rigorous, but technical introduction to epidemiological modelling see[2]). These models are instrumental in fighting epidemics, for example by quantifying the dynamics of diseases, by predicting how epidemics will spread in different contexts, and by providing ways to explore the effects of different possible scenarios of eradication.

[1] W.O. Kermack and A.G. McKendrick. A contribution to the mathematical theory of epidemics. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 115:700–721, 1927. DOI: 10.1098/rspa.1927.0118

[2] O. Diekmann and J.A.P. Heesterbeek. *Mathematical Epidemiology of Infectious Diseases*. John Wiley and Sons Ltd. Chicester, England, 2000

Note that applications are not limited to human diseases. For instance, epidemiological models also play a very important role in agriculture.

The Kermack-McKendrick model considers different types of individuals, such as susceptibles, infected, and removed ones. In this context 'removed' may mean recovered from the disease, or deceased. It is a dynamical model, that follows the changes in the (relative) numbers of these different types of individuals in the course of time. Model parameters are for instance contact- and infection- rate, and the rate of removal. In the original model, total population size was held constant. Later, the model was generalised in many different ways, for instance to include births and migration, additional types of individuals, such as those that are infected but not (yet) infectious, and population structure, such as age, sex, location, or relatedness.

7.1 The SIS model

The original Kermack-McKendrick model is a continuous-time model, consisting of a system of differential equations, and so are many of its generalisations. In this section we will consider a discrete-time version of the SIS model. Note that there are is also a continuous time version of this model. In this model there are two types of individuals: those who are susceptible, and those who are infected. The name *SIS models*, refers to 'Susceptible- Infected - Susceptible'.

The background theory for this section is provided in chapters 9 and 10.
Although the model is initially formulated in a multivariate form, the assignments in this section can all be answered with the theory of univariate models. Subsection 11.1.1, explains the notation for a two-variate model.

1. Model formulation and parameters
   The number of susceptibles at time n is denoted by $S(n)$, and the number infected by $I(n)$. Susceptible individuals may become infected by contacts with infected ones. In this model it is assumed that infected individuals recover at a constant rate, after which they become susceptible again.

   a. Give an example of a disease that might be modelled in this way, and give an example of a disease that cannot. Motivate your answer.

   b. The total population size, N, is assumed to be constant. This means, that the disease dynamics, which are determined by infection and recovery rate, are assumed to be fast compared to the population dynamics, determined by the birth- and mortality rate. Give an example of an infectious disease where this is a reasonable assumption, and give an example of an infectious disease where this is not.

      We assume that the population is well-mixed, and the probability per time step that a susceptible individual gets infected is equal to $\frac{\beta}{N}I(n)$. The recovery rate is denoted by γ. The model equations are:

$$\begin{aligned} S(n+1) &= S(n) - \frac{\beta}{N} \cdot S(n) \cdot I(n) + \gamma \cdot I(n) \\ I(n+1) &= I(n) + \frac{\beta}{N} \cdot S(n) \cdot I(n) - \gamma \cdot I(n) \end{aligned} \tag{7.1}$$

   c. The number of individuals that become infected in year n cannot be larger than the number of individuals that are susceptible in that year. Give an inequality for the parameter β, derived from this constraint.

   d. Use the fact that $I(n)$ is at most equal to N to derive a condition for β that guarantees that this constraint is satisfied.

e. Formulate a similar constraint for the parameter γ, and give the ensuing inequality.

f. Given their interpretation, what other constraint must hold for both parameters?

2. The *basic reproduction rate* or *basic reproductive ratio* R_0 and parameter estimation.

a. The parameter γ is the rate of recovery per time step. This means that the expected duration of the infectious period is $\frac{1}{\gamma}$ time steps. What is the dimension of γ?

b. For influenza, the infectious period of adults is from one day before the onset of symptoms until 5 days after, so about 6 days. Use this to estimate γ for this disease, if we take days as a time unit.

This information, and similar statistics for other diseases can be found on the CDC (Centers for Disease Control and Prevention) website: `https://www.cdc.gov`: just type 'influenza' in the search field.

The basic reproduction rate R_0 is defined as the expected number of infections that are caused by one infected individual if the whole population is susceptible (i.e. in the initial stages of a disease). This quantity has been measured and recorded for many infectious diseases, although it depends on the specific conditions in a population, and it is controversial for diseases with complex dynamics. If the dynamics are described relatively well by the model considered here, R_0 equals $\frac{\beta}{\gamma}$.

With more complex models, it is not so clear how R_0 should be defined and calculated, and its usefulness is debated. In practice several, different measures are used to determine the risk that a disease will spread.

c. What is the (expected) number of new infections at time n?

Look at the model equations in (7.1).

d. What is the value of this quantity if there is only one infected individual, and the number of susceptibles is (approximately) N?

e. Explain that in this case $R_0 = \frac{\beta}{\gamma}$ is indeed in agreement with its definition.

f. One recorded value of R_0 for influenza is about 2.5. Use this to estimate the value of β for this disease (round off to two decimals).

g. The value of $\frac{\beta}{N}$ may be interpreted as the probability that a susceptible gets into contact with an infected individual and contracts the disease, just after introduction of the disease in the population. This is the product of the probability of contact between a susceptible and an infected individual at this stage, and the probability of transmission given contact. Suppose that this probability is about 50% for influenza. Give an estimate for the probability of contact (just after introduction of the disease) in a population of size N.

3. Equilibria

   a. Show that the population size indeed remains constant in this model.

   b. The assumption that population size is constant implies that $S(n) + I(n) = N$. Use this to derive a univariate model for the relationship between $I(n+1)$ and $I(n)$.

   c. Show that $\hat{I}_1 = 0$ is an equilibrium of this model.

   d. This equilibrium is called the 'trivial equilibrium', because any reasonable model for epidemics needs to have this equilibrium value. Why is this so?

   e. This model may have one other equilibrium. Which?

   f. An equilibrium is only relevant in an epidemiological context when it is larger than or equal to zero. For which parameter combinations is the second equilibrium larger than zero?

   g. How does this result relate to R_0?

Derive an equation for this equilibrium.

In model terminology: this equilibrium *exists* only for these parameter combinations.

4. Stability of equilibria

   a. For which parameter combinations is the trivial equilibrium stable?

   b. What does it mean in practice if this equilibrium is stable?

   c. Examine when the other equilibrium is stable, under the condition that it exists.

   d. Show that $\beta - \gamma > -2$ and $\gamma - \beta > -2$.

   e. Summarise your conclusions concerning the stability and existence of the equilibria in relation to the parameters β and γ, based on all previous results.

   f. Formulate the conclusions in epidemiological terminology.

   g. Examine the validity of your conclusions numerically, using Excel, R, or other means.

Methods for doing this are described in subsection 10.4.3.

Use the conditions on the parameter values, that you have derived in assignment 1.

5. Disease mitigation strategies: vaccination
   One of the possible strategies to prevent a disease from spreading in a population is vaccination. Models can be used to determine which part of the population should be vaccinated.

   a. Suppose that a fraction p of the population is vaccinated, before the introduction of an infection. What is the initial number of susceptible individuals in that case?

   b. As before, denote the number of susceptibles after the introduction of an infection by $S(n)$ and the number of infected individuals by $I(n)$. Express $S(n)$ in terms of N, p, and $I(n)$.

   c. What is the recurrence relation for $I(n)$ in this case?

   d. For which parameter combinations is the trivial equilibrium in this model stable?

   e. For which parameter combinations is there a second equilibrium, and what is its value?

f. When is the second equilibrium stable?

g. What proportion of the population should be vaccinated to prevent the spread of a disease like influenza, according to this model?

h. The previous calculations assume that a vaccine is 100% effective. The effectiveness of influenza vaccination, however, is typically only about 60%. How does this affect the conclusions?

6. Alternative model formulations: different infection parameter definition
Models such as the one we have examined are sometimes represented in a different way in the literature. It is important to be aware of this, since it affects the way the model looks, and the interpretation of results, and its parameters.

   a. Sometimes, an alternative definition of the infection parameter is used, namely a single symbol instead of $\frac{\beta}{N}$. To examine the consequences, substitute $b = \frac{\beta}{N}$ in (7.1). What are the new model equations?

   b. What is the expression for R_0 with this formulation?

   c. What is the expression for the 'non-trivial' equilibrium, and its condition for existence with this model formulation?

   d. What is the condition for stability of the trivial equilibrium?

7. Alternative model formulations: proportions instead of numbers
Sometimes models are formulated using proportions of susceptibles and infecteds as state variables rather than absolute numbers.

   a. Transform the model equations in (7.1) to proportions, by dividing by N on both sides of the equations.

   b. Define new state variables: $x(n) = \frac{S(n)}{N}$ is the proportion of susceptibles, and $y(n) = \frac{I(n)}{N}$ the proportion of infected individuals. Give the recurrence relations for $x(n)$ and $y(n)$.

   c. What relation must hold between $x(n)$ and $y(n)$ if the population size is constant?

   d. Use this to derive a univariate model for $y(n)$.

7.2 The SIR model

Diseases like measles provide life time immunity to those who survive. To model such situation, three types of individuals are distinguished: susceptibles, infecteds, and removed. The 'removed' individual may be interpreted as having recovered and immune, or being deceased. These type of models are called SIR models, after the three types considered. The model we will consider here is the classical Kermack-McKendrick model. It is a continuous-time model, specified by a system of differential equations.

The background theory for this section is provided in chapters 9 and sections 12.1,12.2, 12.3, 13.1.1, and 13.2.1. Although this epidemiological model is a multivariate continuous time model, due to its special nature most of the methods in chapter 13 are not useful for its analysis.

1. Model formulation and parameters
   We consider three state variables: the number of susceptible individuals, who have never been infected yet $S(t)$, the number of infected (and infectious) individuals $I(t)$, and the number of removed individuals $R(t)$.

   a. In this model it is assumed that the total population size N, i.e. the sum of $S(t), I(t)$ and $R(t)$, remains the constant. This means that the disease dynamics, which are determined by infection and recovery rate, are assumed to be fast compared to the population dynamics, determined by the birth- and mortality rate. Give an example of an infectious disease where this is a reasonable assumption, and give an example of an infectious disease where this is not.

      It is assumed that the population is well-mixed, implying that every individual has the same contact rate with others, and that there is a constant infection rate α and a constant recovery rate γ. The model equations are:

The fact that we assume that α is constant means that 'removed' should be interpreted as immunity, not mortality, because the contact rate is affected by the total number of mobile individuals.

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= -\alpha \cdot S \cdot I \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \alpha \cdot S \cdot I - \gamma \cdot I \\ \frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma \cdot I \end{aligned} \tag{7.2}$$

   b. Show that the sum of $S(t), I(t)$ and $R(t)$ does not change in time.

This means that the derivative of their sum is zero.

   c. If time is measured in days, the unit of $\frac{\mathrm{d}S}{\mathrm{d}t}$ is 1/days. What are the units of the other differentials?
   d. What are the units of the parameters α and γ?
   e. What ranges of values can the parameters have?

2. The *basic reproduction rate* R_0 and parameter estimation.

   a. The parameter γ is the rate of recovery per time step, so the expected duration of the infectious period is $\frac{1}{\gamma}$ time units. For measles, the infectious period is on average from 4 days before to 4 days after rash onset, so about 8 days. Use this to estimate γ for this disease, if we take days as a time unit.

This information, and similar statistics for other diseases can be found on the CDC (Centers for Disease Control and Prevention) website: `https://www.cdc.gov`: just type 'measles' in the search field.

      The basic reproduction rate R_0 is defined as the expected number of infections that are caused by one infected individual during the infectious period, if the whole population is susceptible (i.e. in the initial stages of a disease). This quantity has been measured and recorded for many infectious diseases, although it depends on the specific conditions in a population, and it is controversial for diseases with complex dynamics. If the dynamics are described relatively well by the model considered here, R_0 equals $\frac{\alpha \cdot N}{\gamma}$.

b. What is the (expected) number of new infections per time unit *per infected individual*?

Look at the first model equation in (7.2).

c. What is the value of this quantity if the number of susceptibles is (approximately) N?

d. Explain that in this case $R_0 = \frac{\alpha \cdot N}{\gamma}$. is indeed in agreement with its definition.

e. One recorded value of R_0 for measles is about 17. Use this to estimate the value of $\alpha \cdot N$ for this disease (round off to two decimals).

f. The probability that a susceptible gets into contact with an infected individual and contracts the disease within a short time period Δt, just after introduction of the disease in the population equals $\alpha \cdot \Delta t$.

The parameter α is the product of contact rate between a susceptible and an infected individual at this stage, and the probability of transmission given contact. This latter probability is quite high with measles, namely about 90%. Give an estimate for the contact rate in a population of size N.

3. Initial disease dynamics

a. At the initial stage, just after an infection has been introduced in a population, $S \approx N$. Use this to derive an approximate univariate model for the change in I at this stage.

b. What is the initial growth rate of the disease?

c. What is the equilibrium value of this model, and when is it stable?

See section 12.3.

d. Epidemics are predicted to occur when $R_0 > 1$ and not to happen when $R_0 < 1$. Argue, on the basis of your results, that this is indeed a valid prediction.

4. Disease mitigation strategies: vaccination
One of the possible strategies to prevent a disease from spreading in a population is vaccination. Models can be used to determine which part of the population should be vaccinated.

a. Suppose that a fraction p of the population has been vaccinated, before the introduction of an infection. What is the initial number of susceptible individuals in that case?

Note that we are only considering the initially susceptible population. If measles has occurred in the population before, there may also be individuals that have already acquired immunity.

b. What is the initial growth rate of the disease in this situation?

c. How large should p be to prevent an epidemic of a disease like measles?

5. Disease dynamics after the initial stage

This part is quite technical.

a. Argue that to study the dynamics, the value of R does not matter in this model, and we might consider it as a two-dimensional system.

b. Use the chain rule to derive an expression for $\frac{dI}{dS}$

Divide the second differential equation in (7.2) by the first one.

c. If all is well, you get an equation of the form $\frac{dI}{dS} = f(S)$. This equation may be solved by the technique of 'separation of variables', which works as follows:

$$\int dI = \int f(S)\, dS \Rightarrow I = F(S) + C$$

where $F(S)$ is the antiderivative of $f(S)$. Use this to show that:

$$I = \frac{\gamma}{\alpha} \ln(S) - S + C$$

where C is a constant (which is determined by the initial conditions).

In section 12.2 it is explained how to do this for a linear equation.

d. In the limit, if t approaches zero, I tends to zero, and S tends to to N. Use this to find an (approximate) value for the constant C.

e. Recall that for measles, R_0 is approximately 17. Use this to make a phase plot of I versus S for different population sizes N.

f. Show that the peak of the epidemic occurs at a value of $S = \frac{\gamma}{\alpha}$.

g. What is the peak number of infected individuals for measles in a population of 1000 individuals?

6. Alternative model formulations: different infection parameter definition

   The SIR model and its generalisations are sometimes represented in a different way in the literature. It is important to be aware of this, since it affects the way the model looks, and the interpretation of results, and its parameters.

   a. Sometimes, an alternative definition of the infection parameter is used, namely $\beta = \alpha \cdot N$. To examine the consequences, adjust the first two model equations in (7.2) accordingly.

   b. What is the expression of R_0 in terms of β?

   c. For which values of β does the model predict that there will be an epidemic?

   d. What is the estimate of β for measles, based on the previous information?

7. Alternative model formulations: proportions instead of numbers

   Sometimes models are formulated using proportions of susceptible, infected, and recovered individuals as state variables rather than absolute numbers.

   a. Transform the model equations in (7.2) to proportions, by dividing by N on both sides of the equations.

   b. Define new state variables: $x = \frac{S}{N}$ is the proportion of susceptibles, $y = \frac{I}{N}$ the proportion of infected individuals, and $z = \frac{R}{N}$ the proportion of recovered individuals. Give the differential equations for these state variables.

   c. In this type of model formulation, usually $\beta = \alpha \cdot N$ is used rather than α. Adjust the equations accordingly.

d. What relationship holds for the state variables if the population size, N, is constant?

e. What does this imply for the differential equations?

8
Pathogen dynamics

Within-host dynamics between pathogens, such as viruses, and the immune system involve many different components and mechanisms. Mathematical models provide insight in the progression of diseases in the host body. They help to generate and test hypotheses concerning processes that occur within the host body, suggest experiments, and develop therapeutic measures. Models such as these were, for instance, instrumental in uncovering the mechanisms through which HIV attacks the host body, and, thereby, to increase effectiveness of treatments[1].

[1] D. Wodarz and M.A. Nowak. Mathematical models of hiv pathogenesis and treatment. *BioEssays*, 24:1178–1187, 2002

8.1 A basic model for virus dynamics

The background theory for the assignments in this section is given in chapters 9, 12 and 13.

This model captures the basic mechanisms involved in within-host virus dynamics, and serves as a starting point for studies of pathogen-immune system interactions. It is a continuous time model, with three state variables: x, the number (or density) of uninfected cells, y the number of infected cells, and v, the number of free virus particles in the body. It is assumed that uninfected cells are produced at a constant rate λ and die at a constant rate μ. They are turned into infected cells at a rate that is proportional to the number of viruses. Infected cells are produced when uninfected cells become infected, and are assumed to die at a constant rate α. Virus particles are assumed to be produced by the infected cells at a constant rate κ, and are removed from the system at a constant rate ρ. This leads to the following model equations:

$$
\begin{aligned}
\frac{dx}{dt} &= \lambda - \mu \cdot x - \beta \cdot x \cdot v \\
\frac{dy}{dt} &= \beta \cdot x \cdot v - \alpha \cdot y \\
\frac{dv}{dt} &= \kappa \cdot y - \rho \cdot v
\end{aligned}
\tag{8.1}
$$

1. Model parameters

   a. What is the meaning of the parameter λ and which values can it assume?

   b. The parameter β is the rate at which an infected cell infects uninfected ones. What is its dimension, and which values can it assume?

c. Give the units of all other parameters, if time is measured in days. What values can they assume?

d. In this model, the (expected) life time of an infected cell is $1/\alpha$. The *burst size* is the total (expected) amount of virus particles that is produced by one infected cell during its lifetime. What is the burst size in this model?

e. What is the expected life time of a free virus particle?

2. The basic reproductive ratio, R_0
The basic reproductive ratio is defined as the number of newly infected cells that arise from one infected cell just after an infection, when almost all host cells are uninfected. It is a crucial quantity that determines whether or not the virus will spread in the host body. If $R_0 < 1$ the virus will be removed successfully by the immune system, if it is larger than one, the virus will succeed in invading the body. For the current model, the value of this quantity equals:

The same concept is used in the context of epidemics.

$$R_0 = \frac{\beta \cdot \kappa \cdot \lambda}{\alpha \cdot \rho \cdot \mu} \tag{8.2}$$

a. What are the initial values of y (y_0) and v (v_0) at the time when infection occurs?

b. In the situation before infection, what is the equilibrium number of uninfected cells? This is the initial value of x, x_0.

c. If there would be just a single infected cell, what would be the equilibrium number of virus particles?

d. In that case, assuming that x is constant at the value x_0, what is the rate at which new infected cells are produced?

e. Explain why the expression for R_0 given in (8.2) is indeed correct for this model.

Initially, the dynamics are governed by the two-dimensional system of differential equations:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \beta \cdot \frac{\lambda}{\mu} \cdot v - \alpha \cdot y \\
\frac{\mathrm{d}v}{\mathrm{d}t} &= \kappa \cdot y - \rho \cdot v
\end{aligned} \tag{8.3}$$

f. What is the equilibrium value?

g. What happens when this equilibrium is stable, and what happens when it is unstable?

h. This is a model of the form given in Eq. (13.3). What is the matrix $\mathbf{A}$ in this case?

i. What is the trace of the matrix?

j. What is its determinant?

k. Show that the equilibrium is stable if and only if $R_0 < 1$.

3. The course of an infection
   If the virus succeeds in invading the body, its dynamics are determined by (8.1). This is a nonlinear system of differential equations, with two equilibrium values. The *trivial equilibrium*, where there is no virus, which we have considered above, and a positive equilibrium that corresponds to a stable focus.
   a. Choose parameter values for which $R_0 > 1$ and examine the course of an infection numerically, starting with $x(0) = x_0, v(0) = 0.01, y(0) = 0$.
   b. Show algebraically that the non-trivial equilibrium value is:
   $$\hat{x} = \frac{x_0}{R_0}, \hat{y} = (R_0 - 1) \cdot \frac{\mu \cdot \rho}{\beta \cdot \kappa}, \hat{v} = (R_0 - 1) \cdot \frac{\mu}{\beta} \qquad (8.4)$$
4. Effects of antiviral therapy
   Drugs like *reverse transcriptase inhibitors* prevent virus particles from infecting new cells, i.e. by reducing the value of β.
   a. What is the value of β for a drug that is 100% effective?
   b. Adjust the model accordingly.
   c. What are the initial values for y and v, assuming that the infection is well on its way at the start of the drug therapy? We will denote these values y^* and v^*.
   d. Solve the differential equation for y, and find an expression for $y(t)$. What does a graph of this function look like?
   e. The *half-life* of the virus producing cells is the time at which the original value of y has decreased by a factor 1/2. What is this value?

   When ρ is large compared to α, the number of free virus particles, $v(t)$, is approximately:
   $$v(t) = \frac{v^*}{\rho - \alpha} \cdot \left(\rho \cdot e^{-\alpha \cdot t} - \alpha \cdot e^{-\rho \cdot t}\right) \qquad (8.5)$$
   f. Show that this is a decreasing function of t. What is the (approximate) decay rate for large values of t?
   g. Show that this function is initially concave
   h. Sketch a graph of the function $v(t)$.

   Other drugs, such as *protease inhibitors* prevent infected cells from producing infectious virus particles. Free virus particles, that were produced before therapy, will for a while continue to infect new cells.These particles are denoted by w.
   i. Give a model for the dynamics of y, v, and w, assuming that the uninfected cell population is approximately constant and equal to x^*.

You should arrive at a model with only two differential equations.

The function $v(t)$ has an initial 'shoulder phase', with a duration of approximately $1/\rho$, after which is also decays exponentially at the same rate as y.

8.2 *HIV dynamics*

Models such as those considered in the previous section show predicted responses to anti-viral therapy that are s-shaped. The

The background theory for the assignments in this section is given in sections 9.1, 9.2, and chapters 12 and 13.

initial duration of the 'shoulder' relates to drug efficacy, mortality rate of infected cells, α, and viral decay rate, ρ. The final decay rate equals α. Treatment of HIV infection with such therapies indeed shows an initial shoulder in virus load, followed by an 'exponential decay' phase with a half life of about one to three days. The estimated half life of free virus particles in the initial phase is of the order of a few hours or less. After the initial, fast decline, however, the decrease in virus load slows down considerably. More importantly, cells can become latently infected with HIV, and have a very long life-span. Complete virus eradication is normally not possible, due to such cells.

An important factor in HIV infection is the virus' ability to diversify. This allows it to attack a large range of target cells, and induces an 'arms race' with the host immune system. Models of the evolution of this *antigenic escape* are used to study the mechanism of HIV disease progression.

The following model, developed by Nowak et al.[2], considers several populations of virus strains, with population sizes $v_i (i = 1, \ldots, n)$ that are generated by mutation (so the number n of strains may vary in time), together with immune responses that each target a specific virus strain. These are modelled by n state variables r_i that represent for instance the density of antibodies that are specifically directed against virus strain i. Furthermore, the model contains a 'group specific' immune response, consisting of the density of antibodies z, that are aimed at the total virus population:

[2] M.A. Nowak R.M. Anderson A.R. McLean T.F.W. Wolfs J. Goudsmit R.M. May. Anitgenic diversity thresholds and the development of aids. *Science*, 254(963-969), 1991; and M.A. Nowak et al. Antigenic oscillations and shifting immunodominance in hiv-1 infections. *Nature*, 375(606-611), 1995

$$\begin{aligned}\frac{\mathrm{d}v_i}{\mathrm{d}t} &= v_i \cdot (a - b \cdot r_i - c \cdot z) \\ \frac{\mathrm{d}r_i}{\mathrm{d}t} &= f \cdot v_i - g \cdot r_i - h \cdot v \cdot r_i \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= m \cdot v - g \cdot z - h \cdot v \cdot z \end{aligned} \tag{8.6}$$

where v denotes the total virus population:

$$v = \sum_{i=1}^{n} v_i \tag{8.7}$$

The summation notation is explained in section 16.1.

The parameter a denotes the rate of virus replication of all strains, g is the decay of antibodies, and h the ability of the virus to impair the immune responses. b and c are the efficacies of respectively the specific and the general immune responses, and f and m the rates at which they are evoked.

1. The total virus load
   The time scales at which the immune responses operate are usually fast compared to those at which the virus population changes.

   a. Assume that the v_i are constant, and calculate the *steady state* values of r_i and z. We will denote these by $r_i{}^*$ and z^*.

b. Show that

$$\sum_{i=1}^{n} v_i \cdot {r_i}^* = \frac{f \cdot v^2}{g + h \cdot v} \cdot D,$$
$$\text{with } D = \sum_{i=1}^{n} \left(\frac{v_i}{v}\right)^2 \tag{8.8}$$

2. The Simpson index of diversity.
   D is called the *Simpson index*. It is a measure of viral diversity.

   a. What is the value of D if there is only one virus strain? What is its value if there are three strains, all with equal abundance? Does D increase or decrease with diversity?

   b. Which values can D assume? For which value is diversity maximal?

   c. What is the value of D just after the initial infection, and what happens to its value in the course of time?

Infection is caused by just a few virus particles.

3. The steady state virus load
   Assuming that the immune responses are in steady state, we now turn to the dynamics of the total virus population.

   a. Show that this gives the approximation:

$$\frac{dv}{dt} \approx v \cdot \left(a - \frac{b \cdot f \cdot v}{g + h \cdot v} \cdot D - \frac{c \cdot m \cdot v}{g + h \cdot v}\right) \tag{8.9}$$

Recall that $v = v_1 + \cdots + v_n$ and use the sum rule for derivatives.

   b. There are two steady states for v, the trivial one, where $v = 0$, and another one at $v = v^*$. Show that the trivial steady state is unstable.

   c. Show that the other steady state v^* is stable, as long as it is positive.

Use the equilibrium condition to simplify the mathematical equations.

   d. Show that:

$$v^* = \frac{a \cdot g}{b \cdot f \cdot D + c \cdot m - a \cdot h} \tag{8.10}$$

4. Progress of the disease
   What happens after infection depends on the virus replication rate a and its rate of impairment of the immune system h, relative to the effectiveness of the group- and strain-specific immune responses. The value of v^* depends on the Simpson index D, so we will denote it by $v^*(D)$.

   a. What happens with $v^*(D)$ as D increases? What happens when virus diversity increases?

   b. What is the asymptotic value of $v^*(D)$ when virus diversity, and thus the number of strains n, increases indefinitely?

What happens with D?

   c. When this value is positive the total virus load goes to a stable steady state, leading to a chronic infection but no disease. This parameter region corresponds to non-pathogenic SIV infection. For which values of $a \cdot h$ does this happen?

d. What does this mean in terms of virus and immune system characteristics?

   When the asymptotic value of $v^*(D)$ is negative, the immune response is insufficient to suppress the virus. There are several possible situations: the initial value of $v^*(D)$ might be negative right from the start, or it is initially positive.

e. In the first situation, virus replication cannot be controlled at all by the immune responses, and the disease progresses rapidly. For which values of $a \cdot h$ does this happen?

f. In the second situation, the initial value of $v^*(D)$ is positive, but its asymptotic value is negative. In this case the disease is initially suppressed, but as virus diversification progresses, the immune system fails to counter the disease. For which values of $a \cdot h$ does this happen?

g. What does the graph of $v^*(D)$ versus D look like in this situation?

   This last situation corresponds to the development of immunodeficiency disease, where initially virus diversity is low, and the immune system is able to suppress the virus. After a certain threshold diversity value, however, the immune system cannot keep up with the virus anymore, and loses the battle.

h. At which value of D lies the threshold?

Part II

Modelling tools

9 Preliminaries

9.1 What are models for?

A model is a simplified representation of a real world system. The aim of models is to help answer questions concerning that system. The starting point for modelling is, therefore, a practical or scientific question. The nature of the question, i.e. the function of the model, determines which aspects should be included and which not. For example, a blueprint is a model of a building. Its function is to show relevant construction details, such as the location of sockets, pipes, and windows, the surface areas of rooms, their relative locations, etc. It leaves out many other aspects such as the colour of the walls, the material that doors and window sills will be made of, which furniture or plants will be put in the rooms, which paintings on the walls, etc.

The whole body of physics is essentially a well-tested model of reality. Newtonian dynamics are based on a simplified model, that works well for most practical situations. Under more extreme conditions, however, adjustments need to be made, based on Einstein's theories.

There are many reasons why mathematical modelling plays an important role in scientific research. Model formulation enforces a very precise formulation of the relevant assumptions, variables, and parameters in the studied system. The process therefore, often reveals a lack of knowledge of essential aspects of the empirical system, that need further investigation. Furthermore, models serve to interpret empirical results, and often lead to new hypotheses, and suggestion of further experiments. If a model describes certain aspects of an empirical system well, it may be used to do hypothetical experiments, and make predictions. Finally, mathematical and computer models provide ways to test verbal arguments and theories.

It is important to realise that there are different types of models, with different types of goals. *Tactical models* are designed to gain detailed insight in a specific question. For instance, detailed computer models of the human heart are used to examine the causes of different types of heart rhythm irregularities. *Strategic (or conceptual) models* are used to answer a general question, or predict a general pattern. For example, biologists use evolutionary game theory to examine the evolutionary origins of altruism. *Descriptive models* (also called *Black box*, or *statistical* models) are used to control, or predict a process in detail. For instance, multivariate regression models of air pollution levels may be used to predict the pollution

The type of models that we are concerned with in this textbook are mainly strategic models. They are also the basic building blocks of much more complex, tactical models that are used to examine real world systems in detail. The analysis of tactical models usually requires intricate mathematics as well as computer simulations.

level given weather conditions, traffic, industrial activity, etc. The model type determines how its results should be used, and how the model should be evaluated. For instance, whereas a statistical model may be judged on how well its quantitative predictions fit the observed data, the relevance of a conceptual model usually relates to its qualitative outcomes.

An example of a qualitative outcome is that altruism is more likely to occur in situations where nearby individuals are genetically related.

9.2 *Variables and parameters*

Mathematical models are represented by one or more equations that contain *variables* and *parameters*. Whereas variables are quantities that vary, parameters are constants that have not (yet) been specified. For instance, the following model specifies a relation between variables x and y, that depends on one parameter, a:

$$y = a \cdot x \tag{9.1}$$

This model might represent, for instance the expected amount of apples, y that is produced as a function of the amount of trees x in an orchard. The parameter a then is the average amount of apples per tree. Different orchards may have different soil, humidity, amount of sunlight, etc., which lead to different values of a. For each orchard, however, it is a constant, that may be estimated by counting the apple production of a large number of trees.

Model analysis usually involves making predictions about variables, given the parameter values. For instance, the model in (9.1) predicts that, as long as a is positive, the total apple production, y, will increase with the number of trees, x. If, for some reason, a should become 0, there will be no apple harvest. It also states that an orchard with a larger value of a will lead to a larger harvest.

This is of course a very simplistic example, but it hopefully serves to clarify the distinction between variables and parameters.

There are several reasons for using symbols rather than numbers in model specifications. It allows us to derive results that are valid for all, or a wide range of, parameter values. This saves time, since it is not necessary to derive results for every special (combination of) parameter value(s). It also makes the conclusions more generally valid. Furthermore, scientific questions are usually quite general, for instance: under which conditions (for which parameter combinations) will this population go extinct? A final, practical, reason for using free parameters is that in most situations parameter values are not (exactly) known and parameter estimation may be difficult. Usually all we can hope for is to know the range of plausible parameter values.

For instance, mortality or migration rates of wild species under natural conditions are very hard to estimate.

Note that the ranges of values that a variable or parameter may assume are often restricted by their meaning in a practical context. For instance, a proportion can only lie between 0 and 1. Time can only be positive. The same holds for population size, or density. It is important to keep such constraints in mind during the model analysis, since it may render some mathematically correct results meaningless or irrelevant in the context of the model.

9.2.1 *Working with free parameters*

To examine models that contain free parameters, you need to learn to work with equations that contain symbols rather than numbers. Also, it will be necessary to be able to draw and interpret graphs without numbers on the axes.

As an illustration, consider the following equation:

$$y = 2x + 4 \tag{9.2}$$

This equation specifies a linear relation between the variables x and y. Graphically, it is a straight line, with a positive slope equal to 2, and a y-intercept of $y = 4$. The following equation also defines a line with the same y-intercept, but with a negative slope -2:

$$y = -2x + 4 \tag{9.3}$$

In principle we could define an infinite number of straight lines by substituting different numbers for the slope and intercept. The following equation, however, represents all of them:

$$y = a \cdot x + b \tag{9.4}$$

The previous equations are both special cases of this formula: in (9.2), $a = 2$ and $b = 4$, whereas in (9.3) $a = -2$ and $b = 4$.

Whereas the parameters a and b in (9.4) are unspecified, the general shape of a graph of y versus x is the same for all parameter combinations, namely a straight line. By substituting numbers for the parameters, specific lines may be defined. If you want to make a graph of y versus x with numbers on the axes, you must choose specific numerical values for the parameters. To say something about the general shape of the graph, however, this is not necessary. For instance, no matter what the value of a is, as long as it is negative, the line will slope downwards. If b is positive, it will cross the $y - axis$ at a point above the origin. If we do not put numbers on the axes, this general shape is as represented in Fig. 9.1.

In this figure we have also indicated the $x - intercept$. To determine this we need to find the value of x at which $y = 0$, so it involves solving a linear equation. To do this, treat a and b as any other constant:

$$\begin{aligned} a \cdot x + b &= 0 \\ a \cdot x &= -b && \text{(subtract } b \text{ from both sides)} \\ x &= -\frac{b}{a} && \text{(divide by } a \text{ at both sides), condition: } a \neq 0 \end{aligned} \tag{9.5}$$

The only difference with working with numbers is that care should be taken not to divide by zero. Therefore, a condition has to be imposed on a in the last step.

This section contains a brief introduction. Techniques for solving equations and inequalities with free parameters are further explained in chapter 18. To learn how to draw graphs of basic functions, study chapter 17. For more complex functions you may need to examine a function's shape by means of its derivatives, see chapter 19.

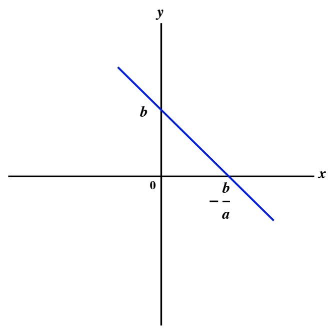


Figure 9.1: the graph of (9.4) when $a < 0$ and $b > 0$.

Exercises and assignments

1. a How does the graph of y in the example of Eq. (9.4) look when $a = 0$?

   b What is the solution x of the equation $a \cdot x + b = 0$ when $a = 0$?

9.3 Units and dimensions

In practical applications of models it is important to be aware of which units of measurement are being used for variables and parameters, since the interpretation of a model's outcome and its consequences depends strongly on this. The *SI system* of standardised units specifies the seven basic units of measurement. It is the most widely used system, and most scientific journals require that authors use this. However, in practice other units are often used, especially on websites. For instance, the SI-unit of temperature is degrees Kelvin, but many people still use Celsius or Fahrenheit, also in model formulations. The *physical dimension* of a quantity is its type, such as mass, length, time, etc. These are examples of *basic dimensions*. There are seven basic physical dimensions. For each of them an SI unit has been specified. The basic dimensions are listed in Table 9.1, together with their corresponding SI units and their standard notations.

The abbreviation SI stands for Système international d'unités.

See for instance `https://www.csee.umbc.edu/portal/help/theory/units.shtml`.

Quantity (dimension)	**SI unit**	**SI unit notation**
length	metre	m
mass	kilogram	kg
time	second	s
temperature	Kelvin	K
luminous intensity	candela	cd
electrical current	ampère	A
amount of a substance	mole	mol

Table 9.1: Basic physical quantities and their units

Note that the mole, although included in this table, is not a physical dimension.

9.3.1 Composite physical quantities

All other physical quantities and their units are derived from those listed in Table 9.1. For instance, velocity is an example of a composite physical quantity, consisting of a combination of length and time. Its dimension is $\frac{\text{length}}{\text{time}}$, and its SI units are metre/second.

An often used unit of energy is the Joule. Expressed in SI units, one Joule equals one kg m^2s^{-2}. From these units it can be seen (cf. Table 9.1) that the dimension of energy is mass $\cdot$ length$^2 \cdot$ time^{-2}.

Working with composite dimensions and units often involves handling powers. See section 17.4.

Power is often expressed in Watts. One Watt equals one Joule per second. The dimension of Watt is energy/time. Expressed in the basic physical quantities listed in Table 9.1) this is mass $\cdot$ length$^2 \cdot$ time^{-3}.

These examples illustrate that when two quantities are multiplied (or divided) their units and dimensions are multiplied (divided) too. In other words, the unit of a product (or ratio) of quantities is the product (ratio) of their units, and the same rule holds for dimensions.

It should also be noted that quantities do not always have a dimension. Quantities such as the numbers of individuals are *dimensionless*. A product or ratio of two quantities that each have a dimension may produce a dimensionless result. For instance suppose that every time unit a constant number of immigrants enters a population. The immigration rate r, the number of individuals that enters the population per time unit, has dimension $1/\text{time}$. The total number of immigrants in a time interval of duration h, $r \cdot h$, which is dimensionless, since h has dimension time.

Exercises and assignments

1. The unit of force is the Newton. Expressed in SI units this is kg m s^{-2}. What is the dimension of force, expressed in the basic physical quantities of Table 9.1?

2. As was stated above, Watt is Joule per second, and, as a consequence, its dimension is energy per time. In a similar way the relation between force and energy may be examined

   a. What is the relation between Newtons and Joules?

   b. Based on this, what is the dimensional relation between force and energy?

3. Acceleration is the increase of velocity per time unit.

   a. What is the (SI) unit of acceleration?

   b. What is the relation between acceleration and force?

4. The *magnetic flux density* is a measure for the strength of a magnetic field. Its unit is the Tesla, which corresponds to $\text{Newton}/(\text{A} \cdot \text{m})$.

   a. What is the dimension of this quantity?

   b. What is its relation with energy?

9.3.2 *Unit conversion*

In practice, data are often not represented in the SI units. Different countries have different conventions concerning the use of units, for instance length is often expressed inches or feet in Britain and the USA. Temperature is expressed in degrees Fahrenheit in these countries, and in degrees Celsius in most European countries. Moreover, different scales may be used in different contexts. For instance, volume may be expressed in cubic centimetres (cm^3) or in cubic metres (m^3). To avoid scaling errors and their consequences, it is important to check the units of measurements, and make sure to be consistent in calculations.

A misplaced decimal point caused a gross overestimation of the iron content of spinach, see e.g.: business insider website.

To execute unit conversions correctly, it helps to write down the units explicitly in the calculations. For instance, 1 mile is approximately 1.609 kilometres. The conversion of 13.3 miles to kilometres, including the units is:

$$13.3 \text{ miles} = 13.3 \text{ miles} \cdot 1.609 \frac{\text{kilometres}}{\text{mile}} \approx 21.4 \text{ kilometres} \tag{9.6}$$

This example may appear trivial, but in more complicated situations such explicit notation makes clear what you are doing, and prevents mistakes. For instance, the slightly more complex calculation of how many miles is 85.3 kilometres goes as follows:

$$85.3 \text{ kilometres} = 85.3 \text{ kilometres} \cdot \frac{1}{1.609} \frac{\text{miles}}{\text{kilometre}} \approx 53 \text{ miles} \tag{9.7}$$

From the units it is clear that now you have to divide by 1.609 rather than multiply. A slightly more complex example involves several unit conversions. Suppose a car travels 25 miles per gallon. What is its fuel economy in kilometres per litre? One gallon is 3.7854 litres, so:

$$\begin{aligned} 25 \frac{\text{miles}}{\text{gallon}} &= 25 \frac{\text{miles}}{\text{gallon}} \cdot 1.609 \frac{\text{kilometres}}{\text{mile}} \cdot \frac{1}{3.7854} \frac{\text{gallons}}{\text{litre}} \\ &\approx 10.6 \frac{\text{kilometres}}{\text{litre}} \end{aligned} \tag{9.8}$$

The same method may be used to transform the scale of the units, for instance when volume is expressed in m^3 rather than cm^3, $1 \text{ m}^3 = 100^3 \text{ cm}^3 = 10^6 \text{ cm}^3$, so:

$$23 \text{ m}^3 = 23 \text{ m}^3 \cdot 10^6 \frac{\text{cm}^3}{\text{m}^3} = 23 \cdot 10^6 \text{ cm}^3 \tag{9.9}$$

Another example is for instance, the conversion of acceleration in kilometres/minute^2, to the SI units, m/s^2:

$$0.02 \frac{\text{km}}{\text{min}^2} = 0.02 \frac{\text{km}}{\text{min}^2} \cdot 1000 \frac{\text{m}}{\text{km}} \cdot \frac{1}{3600} \frac{\text{min}^2}{\text{s}^2} \approx 0.0056 \frac{\text{m}}{\text{s}^2} \tag{9.10}$$

Mistakes are easily made with unit conversions. The method described above is a way to avoid them. An additional way to check whether the conversion is correct is to determine whether the outcome after conversion should be smaller or larger than the original number. For instance, since miles are larger than kilometres, the outcome of a conversion from miles to kilometres should be larger than the original (as in Eq. (9.6)), and the conversion from kilometres to miles should give a number smaller than the original (as in Eq. (9.7)). With more complicated conversions, such as in eqs. (9.8) it helps to do the conversion in steps:

$$\begin{aligned} 25 \frac{\text{miles}}{\text{gallon}} &= 25 \frac{\text{miles}}{\text{gallon}} \cdot 1.609 \frac{\text{km}}{\text{mile}} = 40.225 \frac{\text{km}}{\text{gallon}} \\ 40.225 \frac{\text{km}}{\text{gallon}} &= 40.225 \frac{\text{km}}{\text{gallon}} \cdot \frac{1}{3.7854} \frac{\text{gallons}}{\text{litre}} \approx 10.6 \frac{\text{km}}{\text{litre}} \end{aligned} \tag{9.11}$$

The outcome of the first conversion, from miles to kilometres, should give a larger number. Since litres have a smaller volume than gallons, a car should be able to travel further per gallon than per litre, and therefore the second conversion should result in a smaller number.

Exercises and assignments

1. The density of water is 1 gram/cm^3.How much does 2.51 m^3 of water weigh in kilograms?
2. Every year, about 20 million litres of water are discharged into a river, by a storm drain. The runoff is contaminated: storm drain water contains about 150 micrograms of a toxic heavy metal per litre. How many grams of this metal are dumped into the river per year?
3. A tanker caused a fuel oil spill with a layer of 4.5 inches, over an area of 190 acres. An inch is 2.54 cm and acre is 4.0469 m^2. How many litres of oil were spilled?

metre

9.3.3 *Checking model consistency*

Model equations need to be consistent with respect to units and dimensions. This means that the dimensions and units need to be the same on both sides of model equations. This may be used as an initial test of whether a model is correct.

If a model passes this test it is no guarantee that it is correct. If it fails, however, it is certainly incorrect.

To illustrate this, consider the following equation for the velocity of a falling object. Assuming that, initially, the object has no velocity (it is simply dropped), we have:

$$v(t) = g \cdot t \tag{9.12}$$

where g is the acceleration due to the earth's gravitational force. This equation is dimensionally correct since the dimension of acceleration is velocity/time. Multiplied with time this gives a quantity of dimension velocity. (Expressed in the basic quantities: length/time.) Thus, the dimensions on both sides of the equation are equal.

Expressed in SI units, the value of g is 9.8 m/s^2. If this value of g is used, time is expressed in seconds, and velocity in metres per second, the equation is indeed consistent, since in that case the units on the left hand side are m/s and on the right hand side $(\mathrm{m/s^2}) \cdot \mathrm{s} = \mathrm{m/s}$. Thus, the relation becomes:

$$v(t) \ \text{(in m/s)} = 9.8 \left(\frac{\mathrm{m}}{\mathrm{s}^2}\right) \cdot t \ (\mathrm{s}) \tag{9.13}$$

and, for instance after 2 seconds the velocity of the object is:

$$9.8 \, \frac{\mathrm{m}}{\mathrm{s}^2} \cdot 2 \, \mathrm{s} = 19.6 \, \mathrm{m/s} \tag{9.14}$$

If velocity is expressed in different units, say km/min, time should be expressed in hours on the right hand side, and the value of g

should be expressed in km/min^2:

$$\begin{aligned} 9.8 \text{ m/s}^2 &= 9.8 \text{ m/s}^2 \cdot \left(\frac{1}{1000} \frac{\text{km}}{\text{m}} \right) \cdot \left(60 \frac{\text{s}}{\text{min}} \right)^2 \\ &= \frac{9.8 \cdot 3600}{1000} \frac{\text{km}}{\text{min}^2} \end{aligned} \tag{9.15}$$

Thus, the relation becomes:

$$v(t) \text{ (in km/min) } = 35.28 \left(\frac{\text{km}}{\text{min}^2} \right) \cdot t \text{ (min)} \tag{9.16}$$

and, for instance the velocity after 30 seconds is:

$$35.28 \frac{\text{km}}{\text{min}^2} \cdot 0.5 \text{ min} = 17.64 \frac{\text{km}}{\text{min}} \tag{9.17}$$

Quantities can only be added or subtracted from each other if they have the same units and dimensions. For instance, you cannot subtract a distance from a velocity, or add force to energy. Also, adding 1 km to 1 m does not have an outcome of 2. To get the proper outcome you have to express both quantities either in kilometres or in metres.

This means that if an equation contains several quantities that are added to or subtracted from each other, those quantities should have the same units (and, as a consequence the same dimensions). For instance, a failed rocket launch results in a velocity of:

$$v(t) = v_0 + g \cdot t \tag{9.18}$$

where v_0 denotes the rocket's velocity at the moment of power failure, when it starts dropping back to earth. The value of v_0 is negative, indicating an initial direction away from the earth. Since v_0 is a velocity, it has the same dimension as $g \cdot t$. Since, furthermore, the quantities on both side are velocities, the expression is dimensionally correct. If g and t are expressed in SI units, the initial velocity must also be given in m/s. Otherwise, the equation is incorrect with respect to units.

As an example, suppose that at the moment of power failure the rocket has a velocity of 600 km/hr, in a direction away from the earth. Expressed in km/min the initial velocity is $v_0 = -10$, and we get:

$$v(t) \text{ (in km/min) } = -10 \text{ (km/min)} + 35.28 \frac{\text{km}}{\text{min}^2} \cdot t \text{ (min)} \tag{9.19}$$

Thus, after 5 minutes:

$$-10 \text{ (km/min)} + 35.28 \frac{\text{km}}{\text{min}^2} \cdot 5 \text{ (min)} = 166.9 \text{ (km/min)} \tag{9.20}$$

If a model equation contains a sine, exponential, or logarithmic function, the argument needs to be dimensionless to be interpretable. The reason is that for instance the sine of a metre does not have a meaning. For instance, a model for bacterial growth rate predicts that the bacterial density x (the number of bacteria per litre) at time t equals:

$$x(t) = a \cdot e^{r \cdot t} \tag{9.21}$$

where a and r are positive parameters. The product $r \cdot t$ must be dimensionless. Since t has dimension time, this means that r must have dimension 1/time. Indeed, it turns out that in this model r is a growth rate, with the proper dimension.

Exercises and assignments

1. Show that (9.13) gives the same velocity after 30 seconds as in (9.17).

2. Express the outcome of (9.19) in km/hr.

3. What is the dimension of the parameter a in (9.21)?

4. A model for the thickness h of a slab of ice in the ocean, in relation to temperature states:

$$h(t) = \sqrt{{h_0}^2 + \frac{2k(T_w - T_a)t}{L \cdot D}}$$

   where $h(t)$ is its thickness at time t, h_0 the initial thickness, T_w the water temperature, and T_a the air temperature. k is the thermal conductivity of the ice, D the density of sea water (its weight per volume), and L its *latent heat*. This is the amount of energy that is released per mass when sea water turns into ice.

   a. What is the dimension of $h(t)$?

   b. What must be the dimension of the second expression under the square root sign, for the model to be consistent?

   c. What is the dimension of $L \cdot D$?

   d. What must be the dimension of the heat conductivity, k, to make the model dimensionally consistent?

   e. k is usually expressed in the units $\frac{\text{Watt}}{\text{K}\cdot\text{m}}$. Show that the corresponding dimension agrees with your answer.

10
Deterministic, discrete time, univariate models

A dynamical model describes the change in one or more variables in the course of time. These variables are called *state variables*. Deterministic models assume that, given the current value of the state variables, their future values can be predicted completely, without any uncertainty. There are different types of such models, depending on whether state variables are assumed to change periodically, after fixed time steps, or continuously, at any moment.

In this chapter we will consider a selection of models chosen on the basis of their usefulness for demonstrating different concepts, techniques, and dynamical phenomena. There are many more models, of varying complexity and level of realism, but if you understand those treated here, you should be able to understand the structure and analysis of other models too (at least, to a large extent).

10.1 What do they look like?

Discrete time models are specified by *recurrence relations*. An example of such a formula is:

$$x(n+1) = 2 \cdot x(n) \tag{10.1}$$

Here, x is the state variable, and $x(n)$ represents its value at time n. The equation describes the relationship between the value of x at time $n+1$ and that at time n. As soon as $x(n)$ is known, the value of $x(n+1)$ is fixed. For instance, if $x(n) = 1$ the model specifies that the next value of the state variable is 2. If $x(n) = 3$, then $x(n+1) = 6$.

Another example is:

$$x(n+1) = 3 \cdot x(n) - 4 \cdot x(n)^2 \tag{10.2}$$

In this case, if $x(n) = 1$, the value of $x(n+1)$ would be -1. If $x(n) = 3$, $x(n+1)$ equals -27.

Note that some authors call this type of equations *difference equations*. So when you are reading other books, or articles, it is important to determine what terminology people are using. We consider alternative terminology and notations further in the paragraph at the end of this section.

Equations such as (10.1) are sometimes called difference equations.

The general form of a recurrence relation is:

$$x(n+1) = f(x(n)) \tag{10.3}$$

where $f(x)$ is some function. In the example of Eq. (10.1), for instance, $f(x) = 2 \cdot x$. This is an example of a *linear model*, since $f(x)$ is a linear function. The general form of a linear model is:

$$x(n+1) = a \cdot x(n) + b, \tag{10.4}$$

where a and b are the model's parameters.

The example in (10.2) may be called a *quadratic model* because in this case $f(x) = 3 \cdot x - 4 \cdot x^2$, so it has a quadratic form. This is an example of a *nonlinear model*.

Nonlinear models occur in many practical applications. For instance, the following two models are both used in the context of population dynamics, where the state variable represents population density. Both models are named after their inventors.

The Beverton-Holt model[1]:

$$x(n+1) = \frac{a \cdot x(n)}{1 + b \cdot x(n)}, \text{ with } x \geq 0,\ a > 0,\ b > 0 \tag{10.5}$$

The Ricker model[2]:

$$x(n+1) = a \cdot x(n) \cdot e^{-b \cdot x(n)}, \text{ with } x \geq 0,\ a > 0,\ b > 0 \tag{10.6}$$

[1] R.J.H. Beverton and S.J. Holt. On the dynamics of exploited fish populations. *Fishery Investigations Series II, VOlume XIX, Ministry of Agriculture, Fisheries and Food*, 1957

[2] W.E. Ricker. Stock and recruitment. *Journal of the Fisheries Research Board of Canada*, 11(5):559–623, 1954

Alternative notations and terminology

As mentioned, some people refer to equations such as presented above as difference equations. In a more narrow sense, difference equations have a slightly different form: it describes the difference between the state variable at two successive times. The difference equation corresponding to (10.1) is:

$$x(n+1) - x(n) = x(n) \tag{10.7}$$

which is acquired by subtracting $x(n)$ on both sides of (10.1). This equation says that, for instance, if the current value of the state variable $x(n) = 1$, the change in the state variable is 1, so the next value will be 1+1 = 2 just as before. If the current value is $x(n) = 3$, the change in the state variable's value is 3, and the next value is 6. As you can see, the two representations are fully equivalent, and one can be straightforwardly transformed into the other. For the model in (10.2), for instance, the difference equation is:

$$x(n+1) - x(n) = 2 \cdot x(n) - 4 \cdot x(n)^2 \tag{10.8}$$

Sometimes, the left-hand side of a difference equation is represented by a Δ, the greek symbol for D, standing for 'difference', and the dependence on n is indicated by a subscript. The equation in (10.8) would, for instance look like:

$$\Delta(x_n) = 2 \cdot x_n - 4{x_n}^2 \tag{10.9}$$

or

$$\Delta_n = 2 \cdot x_n - 4{x_n}^2 \tag{10.10}$$

In some instances, recurrence and difference equations are written in terms of the relationship between $x(n)$ and $x(n-1)$ rather than $x(n+1)$ and $x(n)$. For instance, (10.1) would then look like:

$$x(n) = 2 \cdot x(n-1) \tag{10.11}$$

Exercises and assignments

1. Consider the general model formulation of Eq. (10.3).
   a. What is the function $f(x)$ for the Beverton-Holt model?
   b. This is one of the basic functions described in chapter 17. Which?
   c. Is the function in the Beverton-Holt model increasing or decreasing in x?
   d. What is the horizontal asymptote of this function?
   e. Sketch a graph of this function.

2. Consider the general model formulation of Eq. (10.3).
   a. What is the function $f(x)$ for the Ricker model?
   b. What is the value of this function at $x = 0$?
   c. What happens to the value of $f(x)$ as x becomes (infinitely) large, and what does this imply for the graph of the function?

      See section 17.8.

   d. This function has one extremum, at which value of x does it occur?

      Calculate the derivative. See chapter 19.

   e. Is this a minimum or a maximum?
   f. What does a graph of this function look like?

3. Write down the general form of a quadratic model. How many parameters are there?

4. Consider the Beverton-Holt model, with $a = 2$ and $b = 1$.
   a. What is the value of $x(n+1)$ if $x(n) = 0.5$?
   b. What is the value of $x(n+1)$ if $x(n) = 2$?
   c. Can you think of a value of $x(n)$ that would give the exactly same value for $x(n+1)$?

5. Consider the Ricker model, with $a = 2$ and $b = 1$.
   a. What is the value of $x(n+1)$ if $x(n) = 1$?
   b. What is the value of $x(n+1)$ if $x(n) = 2$?
   c. Can you think of a value of $x(n)$ that would give exactly the same value for $x(n+1)$?

6. Write the Beverton-Holt model in the form of a difference equation, as was done in (10.8) for the quadratic model.

7. Reformulate the linear model in (10.4) as a recurrence relation between $x(n)$ and $x(n-1)$

10.2 *How do they work?*

Recurrence relations specify an updating rule: when the current value of the state variable is known, the next value can be calculated. Thus, given an initial value $x(0)$, we may calculate $x(1)$. Once $x(1)$ is known, the same procedure can be applied to calculate $x(2)$, and so on. This is represented schematically in Fig. 10.1 for the model in (10.1), and two different values of $x(0)$. As you can see, different values of $x(0)$ lead to different sequences of the state variable. The different outcomes are called *solutions* of the model . The value of $x(0)$ is called the *initial condition* or *starting condition*. Most models have an infinite possible number of solutions, each corresponding to a different initial condition. A recurrence relation together with an initial condition specifies a unique solution.

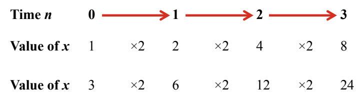


Figure 10.1: Example of the values of the state variable up to $n = 3$, for the model in (10.1), with two different starting conditions.

For some models the relationship between $x(n)$ and n can be specified explicitly. For instance, for the example in (10.1) the pattern can be detected quite easily:

$$\begin{aligned} & x(1) = 2 \cdot x(0) \\ & x(2) = 2 \cdot x(1) = 2^2 x(0) \\ & x(3) = 2 \cdot x(2) = 2^3 x(0) \\ & \vdots \end{aligned} \tag{10.12}$$

So in this case the so-called *solution equation* is:

$$x(n) = 2^n x(0) \tag{10.13}$$

For linear models, solution equations are relatively simple to derive. First, consider a model of the form:

$$x(n+1) = a \cdot x(n) \tag{10.14}$$

Proceeding in the same way as in (10.12) it can be shown that the solution is:

$$x(n) = a^n x(0) \tag{10.15}$$

As you see, the number of time steps, n is the exponent of a in the solution equation. If $a > 1$ this model predicts *exponential growth* of the value of $x(n)$ in the course of time. If $a < 1$ it predicts *exponential decay*. For this reason the model in (10.14) is also called an *exponential growth model*.

Now consider the, more general, linear model:

$$x(n+1) = a \cdot x(n) + b, \text{ with } a \neq 1 \tag{10.16}$$

Subtracting $b/(1-a)$ on both sides and rearranging gives:

It will become clear in later sections why we chose $b/(1-a)$.

$$\begin{aligned} x\left(n+1\right)-\frac{b}{1-a} &= a\cdot x\left(n\right)+b-\frac{b}{1-a} \\ &= a\cdot x\left(n\right)+\frac{b\cdot\left(1-a\right)-b}{1-a} \\ &= a\cdot x\left(n\right)+\frac{-b\cdot a}{1-a} \\ &= a\cdot x\left(n\right)-a\cdot\frac{b}{1-a} \end{aligned} \tag{10.17}$$

so:

$$x\left(n+1\right)-\frac{b}{1-a}=a\cdot\left(x\left(n\right)-\frac{b}{1-a}\right) \tag{10.18}$$

Note that this is again a recurrence relation, between $x(n+1)$ minus the constant value $\frac{b}{1-a}$ and $x(n)$ minus that exact same constant. Thus, if we define:

$$z\left(n\right)=x\left(n\right)-\frac{b}{1-a} \tag{10.19}$$

Equation (10.18) states that:

$$z\left(n+1\right)=a\cdot z\left(n\right) \tag{10.20}$$

From our previous results we can see that:

$$z\left(n\right)=a^{n}z\left(0\right) \tag{10.21}$$

Substitution of the expression for $z(n)$ gives:

$$x\left(n\right)-\frac{b}{1-a}=a^{n}\left(x\left(0\right)-\frac{b}{1-a}\right) \tag{10.22}$$

and rearranging this gives us the solution equation for (10.16):

$$x\left(n\right)=\frac{b}{1-a}+a^{n}\left(x\left(0\right)-\frac{b}{1-a}\right) \tag{10.23}$$

Note that when $b = 0$ this gives the same outcome as in (10.15), as it should.

The linear model is a special case. It is not always possible to find a solution equation. In fact, in all but the simplest models there are no explicit solution equations. For that reason, the dynamics of $x(n)$ are studied numerically, by computing the successive outcomes, as above, and by mathematical techniques that will be explained in later sections. Knowing the solution of the linear model is very useful, since this model may serve as an approximation to other, more complex, models.

Exercises and assignments

1. Consider the model $x(n+1) = 0.2 \cdot x(n) + 1$.

   a. This is a model of the form given in (10.16). What is the value of $b/(1-a)$ in this case?

   b. The solution equation for the general model is given in Eq. (10.23). What is it in this specific case?

   c. What is the value of $x(n+1)$ if $x(n) = b/(1-a)$?
   d. What is the value of $x(n)$ if $x(0) = b/(1-a)$?

2. Consider the model $x(n+1) = 0.5 \cdot x(n) - 2$.
   a. This is a model of the form given in (10.16). What is the value of $b/(1-a)$ in this case?
   b. What is its solution equation?
   c. What is the value of $x(n+1)$ if $x(n) = b/(1-a)$?
   d. What is the value of $x(n)$ if $x(0) = b/(1-a)$?

3. Consider the general expression for the linear model, in Eq. (10.16).
   a. What is the value of $x(n+1)$ if $x(n) = b/(1-a)$?
   b. What is the value of $x(n)$ if $x(0) = b/(1-a)$?

4. Consider the model $x(n+1) = \gamma \cdot x(n) - \beta$.
   a. This is a model of the form given in (10.16). What is the value of $b/(1-a)$ in this case?
   b. What is its solution equation?

5. Calculate the values of $x(1)$ and $x(2)$ for the Beverton-Holt model with $a = 0.2,\ b = 1,\ x(0) = 1$.

6. Calculate the values of $x(1)$ and $x(2)$ for the Ricker model with $a = 0.2,\ b = 0.1,\ x(0) = 1$.

7. Derive the solution equation of the following models, using the same method as in (10.17).
   a. $x(n+1) = 3 \cdot x(n) + 2$
   b. $x(n+1) = 2 \cdot x(n) - 5$

8. Derive the solution of the linear model for the special case that $a = 1$, by writing down the expressions for $x(1), x(2), \ldots$ and finding a pattern, as was done in (10.12). What is the solution equation? What kind of relationship does this model predict between $x(n)$ and n?

10.3 *How do they behave?*

Univariate models can have several different types of dynamics. We will first consider the model $x(n+1) = a \cdot x(n)$. In the previous section it was shown that the solution of this model is $x(n) = a^n \cdot x(0)$. The behaviour of a^n depends on the value of the parameter a. If a is positive, its powers are all positive. If it is negative, they will alternate between positive and negative values. If a is between -1 and 1, the absolute value of a^n will decrease with n, and if it lies outside these bounds, its absolute value will increase. Here are some examples of these different situations:

$$\begin{aligned}
a &= 3, & a^2 &= 9, & a^3 &= 27, \ldots \\
a &= \frac{1}{2}, & a^2 &= \frac{1}{4}, & a^3 &= \frac{1}{8}, \ldots \\
a &= -2, & a^2 &= 4, & a^3 &= -8, \ldots \\
a &= -\frac{1}{2}, & a^2 &= \frac{1}{4}, & a^3 &= -\frac{1}{8}, \ldots
\end{aligned} \tag{10.24}$$

As a consequence, the dynamics of the solution $x(n)$ depend on the values of a:

$|a|$ denotes the *absolute value*, or *modulus* of a. See section 17.2.

- If $-1 < a < 1$, or, in a different notation $|a| < 1$, $x(n)$ converges to zero. It will do so *monotonically* if $0 < a < 1$. If $-1 < a < 0$, the value of $x(n)$ will fluctuate, with decreasing amplitude.

- If $|a| > 1$, $x(n)$ will move away from zero. This will happen monotonically if $a > 1$ and with increasing fluctuations if $a < -1$.

These different types of dynamics are shown in Fig. 10.2 for initial conditions $x(0)$ on either side of 0.

The linear model $x(n+1) = a \cdot x(n) + b$, with $a \neq 1$ has very similar dynamics. In fact, they look exactly the same, but shifted vertically by a distance of $b/(1-a)$. Thus, the dynamics for $x(n+1) = a \cdot x(n) + b$ are as in Fig. 10.2, if we let the horizontal line represent the line $y = b/(1-a)$.

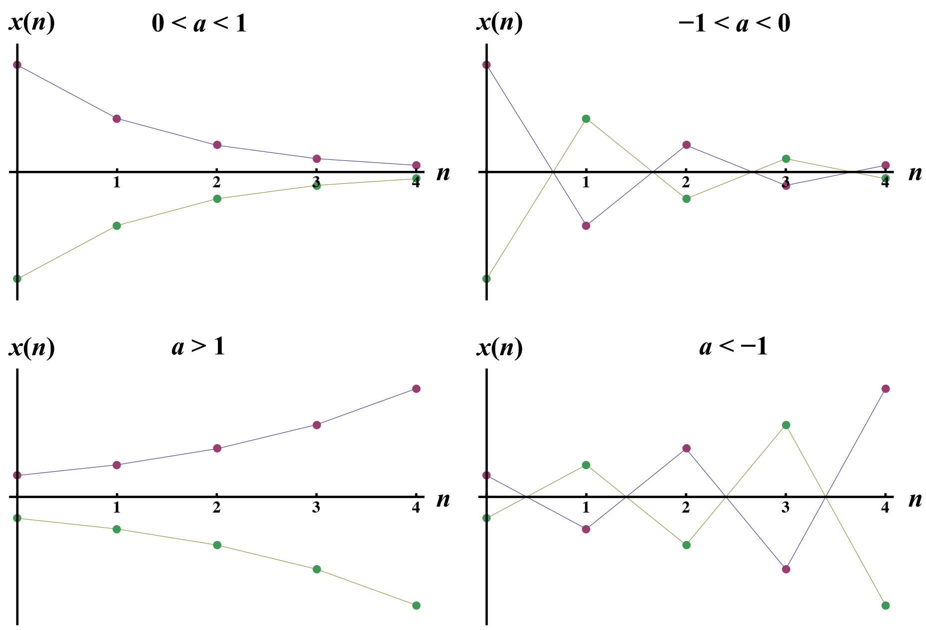


Figure 10.2: Examples of dynamics of the linear model for different ranges of the parameter a.

Nonlinear models sometimes show similar dynamics as the linear model, but they also may show radically different phenomena. To illustrate this, Fig. 10.3 demonstrates different types of dynamics of the state variable for the Ricker model (specified in Eq. (10.6)). The parameter b in this model does not affect the type of its dynamics. As the value of the parameter a in this model increases, the dynamics change from monotonic convergence to zero (see figure at $a = 0.9$), to monotonic convergence to a positive constant ($a = e^{0.1}$), to oscillating convergence to a positive constant (at $a = e^{1.9}$). As a increases further, more exotic behaviour occurs. At $a = e^{2.1}$ converges $x(n)$ to a *stable limit cycle* with a period of 2; at $a = e^{2.6}$ it converges to a stable limit cycle with a period of 4, and at even higher values, such as $a = e^{3.5}$ so-called *chaos* occurs, where $x(n)$ seems to change in a random fashion within a range of values. Keep in mind, however, that this process is fully deterministic.

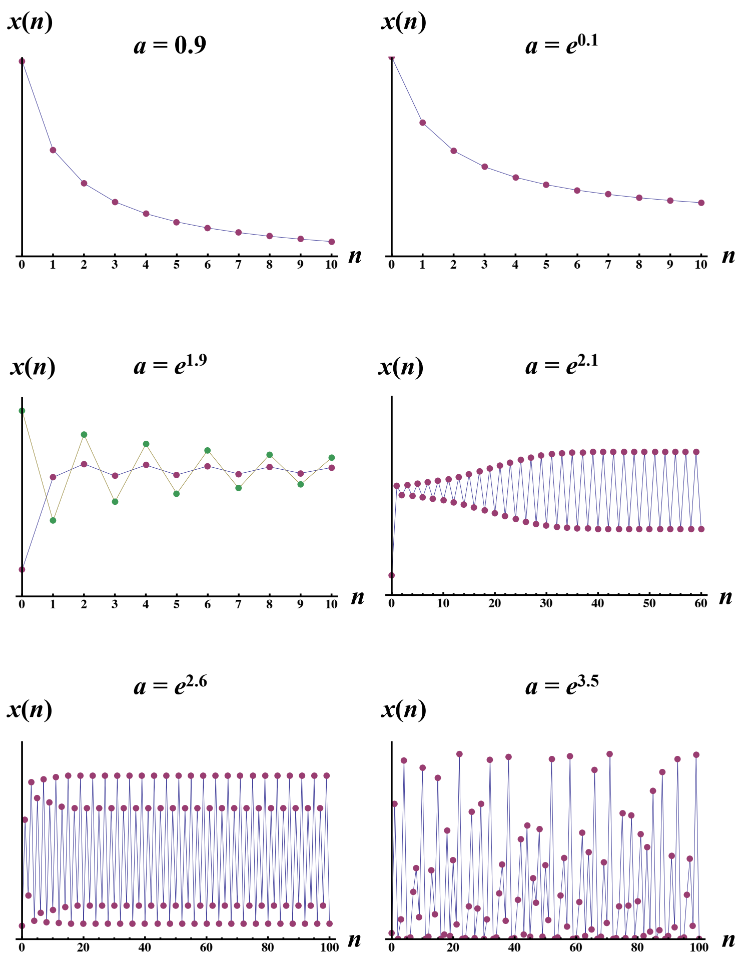


Figure 10.3: Examples of dynamics of the Ricker model for different values of the parameter a.

In the examples shown up to now, the type of long-term dynamics do not depend on the initial conditions, but only on the values of (some of the) parameters. This is not always so. As an example of how initial conditions may affect the dynamics, figure 10.4 shows an example of the dynamics for two different initial conditions of the quadratic model:

$$x\,(n+1) = -x\,(n) \cdot (x\,(n) - 3) \qquad (10.25)$$

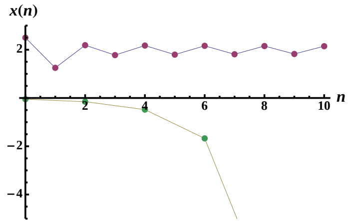


Figure 10.4: Dynamics of the model in Eq. (10.25) for two different initial conditions.

In this model, most initial values between 0 and 2 will lead to a stable limit cycle with period 2, whereas an initial value smaller than zero will eventually lead to infinitely negative values.

The examples in this section demonstrate some important general properties of the dynamics of many deterministic, univariate models:

- There is an initial *transient* phase, where the dynamics are affected by the initial conditions. This initial period may vary in duration, depending on the model and parameter values.
- After the transient phase, the dynamics become *stable*, and do not depend on the initial conditions anymore. .
- Stable dynamics may have different forms: a constant value (which may or may not be zero), a stable limit cycle, or chaos. Such forms are called *attractors* of the system. An attracting, constant value is called a *stable equilibrium*
- Although in most cases the *precise* initial value does not affect long term dynamics, different ranges of initial values may lead to different types of such dynamics.

Exercises and assignments

Use Excel, R, or a graphics calculator for the following assignments.

1. Replicate the different types of dynamics of the Ricker model (see Eq. (10.6)) that were shown in this section. Choose your own value of b.
2. Examine the dynamics of the linear model for different values of a, and a non-zero value of b.
3. Examine the dynamics of the linear model when $a = 1$ and a value of $b \neq 0$.
4. Examine the dynamics of the Beverton-Holt model (see (10.5)). Compare what happens when $a \leq 1$ and when $a > 1$. Also explore the effect of varying b.

10.4 *How can you tell?*

In most applications, the long-term model dynamics are the most relevant, rather than what happens in the transient phase. The examples in the previous session illustrate that there are many different types of long-term dynamics, and that these are determined by parameter values, in some cases in combination with initial conditions. In most situations, however, parameter values and/or initial conditions may vary, and we want to predict what happens for different ranges of those values. It is usually not feasible to explore all possible combinations numerically, especially when models contain many parameters. Fortunately, there are ways to avoid having to do this, and thus perform a more efficient exploration of long-term model dynamics.

10.4.1 Equilibria

The key to the prediction of long-term model dynamics are *equilibrium* points. An equilibrium is a value of the state variable such that $x(n+1) = x(n)$. For instance, the model in Eq. (10.14) as well as the Beverton-Holt model in (10.5), and the Ricker model in (10.6) all have the point 0 as an equilibrium. If the initial value $x(0)$ equals the equilibrium value, the process will remain in the equilibrium, i.e. the state variable will keep the same value as $x(0)$ at all times n.

There are different types of equilibria. *Stable* equilibria are *attractors*. For instance, as mentioned in the previous section, for the linear model $x(n+1) = a \cdot x(n)$, the point 0 is a stable equilibrium if $|a| < 1$. *Unstable* equilibria are *repellers*. The equilibrium point 0 in the linear model is unstable if $|a| > 1$. Note that unstable equilibria are still equilibria: if you choose $x(0)$ exactly equal to an unstable equilibrium, the process will remain there forever. The slightest perturbation, however, will cause the state variable to move away from the equilibrium. Stable equilibria, on the other hand, are robust to slight perturbations. This is illustrated in Fig. 10.5.

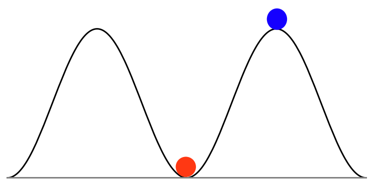


Figure 10.5: Marbles in a hilly landscape with two equilibria: red at a stable equilibrium, blue at an unstable equilibrium.

Knowing the equilibria of a model and their stability allows us to make predictions concerning the direction in which the state variable will move. In some cases, when equilibria are stable, we may even predict where $x(n)$ will end up in the long run. The linear model (10.14) for instance, has only one equilibrium point. When it is stable, the process will move towards that point for all initial values $x(0)$. Note however, that in nonlinear processes there may be several equilibria, and it may depend on initial conditions if a particular stable equilibrium is reached eventually or not.

The first step in a mathematical analysis of a dynamical model is usually to determine its equilibrium points. We will denote equilibrium values by $\hat{x}$, and use subscripts ($\hat{x}_1$, $\hat{x}_2$, etc.) if there are several such points. To find equilibria, fill in $\hat{x}$ for the values of $x(n)$ as well as $x(n+1)$ in the recurrence equation and solve the resulting *equilibrium equation*. For instance, for the general linear model in

(10.16), this gives:

$$\begin{aligned}
&x(n+1) = a \cdot x(n) + b, \text{ with } a \neq 1 && \text{model equation} \\
&\hat{x} = a \cdot \hat{x} + b && \text{equilibrium equation} \\
&\hat{x} - a \cdot \hat{x} = b && \text{subtract } a \cdot \hat{x} \text{ on both sides} \\
&\hat{x} = \frac{b}{1-a} && \text{divide both sides by } 1-a
\end{aligned} \tag{10.26}$$

The Beverton-Holt model in (10.5) has the following equilibrium equation:

$$\hat{x} = \frac{a \cdot \hat{x}}{1 + b \cdot \hat{x}} \tag{10.27}$$

One solution is, obviously, $\hat{x}_1 = 0$. The other is found by dividing both sides by $\hat{x}$ and rearranging:

$$\begin{aligned}
1 &= \frac{a}{1 + b \cdot \hat{x}} \\
1 + b \cdot \hat{x} &= a && \text{multiply both sides by } 1 + b \cdot \hat{x} \\
b \cdot \hat{x} &= a - 1 && \text{subtract 1 from both sides} \\
\hat{x}_2 &= \frac{a-1}{b} && \text{divide both sides by } b
\end{aligned} \tag{10.28}$$

The second equilibrium only exists when $a > 1$, since the model specifies that $x \geq 0$. For other values of this parameter, the model only has one equilibrium, at zero.

The equilibrium equation for the Ricker model (cf. Eq. (10.6)) is:

$$\hat{x} = a \cdot \hat{x} \cdot e^{-b \cdot \hat{x}} \tag{10.29}$$

which has two solutions. The first equilibrium is $\hat{x}_1 = 0$, and the second is found by dividing both sides by $\hat{x}$ and solving the resulting equation:

$$\begin{aligned}
1 &= a \cdot e^{-b \cdot \hat{x}} \\
\frac{1}{a} &= e^{-b \cdot \hat{x}} && \text{divide by } a \text{ on both sides} \\
-\ln(a) &= -b \cdot \hat{x} && \text{take logarithms on both sides} \\
\frac{\ln(a)}{b} &= \hat{x} && \text{divide both sides by} (-\text{b})
\end{aligned} \tag{10.30}$$

so the second equilibrium is $\hat{x}_2 = \frac{\ln(a)}{b}$. This equilibrium only exists when $a > 1$, since in this model the state variable is non-negative. So, this model a single equilibrium at 0 when $a \leq 1$, and it has an additional equilibrium otherwise.

Exercises and assignments

1. Determine the equilibria of the quadratic model in (10.25).

2. How many equilibria can a general quadratic model have at most?

3. Adding proportional harvesting at a rate of h to a Beverton-Holt model gives the following recurrence equation:

$$x(n+1) = \frac{2 \cdot x(n)}{1 + x(n)} - h \cdot x(n), \; x \geq 0, \; a > 0, \; b > 0, \; h \geq 0 \tag{10.31}$$

   a. This model may have at most two equilibria. What are their values?

   b. How large may h maximally be for the second equilibrium to exist?

4. Adding the same type of harvesting to a Ricker model gives:

$$x(n+1) = 1.5 \cdot x(n) \cdot e^{-0.1x(n)} - h \cdot x(n) \tag{10.32}$$

   a. This model may have at most two equilibria. What are their values?

   b. For which range of h-values does the model have two equilibria?

5. Consider the Beverton-Holt model with harvesting, with general parameter values:

$$x(n+1) = \frac{a \cdot x(n)}{1 + b \cdot x(n)} - h \cdot x(n), \; x \geq 0, \; a > 0, \; b > 0, \; h \geq 0 \tag{10.33}$$

   a. What are the expressions for the two equilibria?

   b. Assume that $a > 1$, so there are two equilibria if $h = 0$. How large may h maximally be to keep two equilibria when there is harvesting?

6. Answer the same questions for the Ricker model with harvesting and general parameter values.

10.4.2 Types of equilibria

In the previous sections we have already mentioned that equilibria can be stable or unstable. It should also be noted, however, that there are different kinds of stability.

Equilibria are called *globally stable* if the state variable converges to the equilibrium for every initial value of $x(0)$. An example is the equilibrium 0 in the linear model. In many cases, equilibria are not globally, but *locally stable*. This means that the state variable will converge to the equilibrium if the initial value lies within a certain range, but not if $x(0)$ lies outside that range. The range of initial conditions that lead to the equilibrium is called the *region of attraction* of the equilibrium. These concepts are illustrated in Fig. 10.6

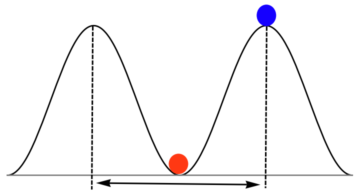


Figure 10.6: Marbles in a landscape, illustrating an unstable equilibrium (blue), and a locally stable equilibrium (red) with its region of attraction.

In most applications we will be concerned with equilibria that are either unstable or (locally or globally) stable, but it should be noted that there are more possibilities. Equilibria are called *neutrally stable* if they are neither attracting or repelling. This concept is illustrated in Fig. 10.7. For example, consider the linear model, with $a = 1$. In that case the model states $x(n+1) = x(n)$, so whatever value of $x(0)$ is chosen, the state variable will stay in that value. This means that any value is an equilibrium for this model, and they are all neutrally stable, because the distance from $x(n)$ to any other point than the initial value neither increases nor decreases in time.

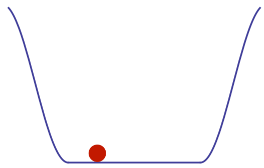


Figure 10.7: A marble in a landscape, at a neutrally stable equilibrium.

Finally, equilibria may be *half stable*, which means that they are attracting from one side and repelling from the other side. Such equilibria are called *saddle points*. This concept is illustrated in Fig. 10.8. As an example, consider the equilibrium $\hat{x} = 0$ in the following quadratic model:

$$x(n+1) = x(n)(1 - x(n)) \tag{10.34}$$

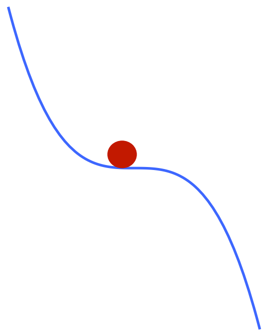


Figure 10.8: A marble in a landscape, at a half stable equilibrium.

If $x(0)$ lies in the interval $[0, 1)$ the state variable converges to the equilibrium 0. If, however, $x(0)$ is smaller than zero, the state variable moves away from 0, no matter how close the initial value lies to the equilibrium. This is demonstrated in Fig. 10.9.

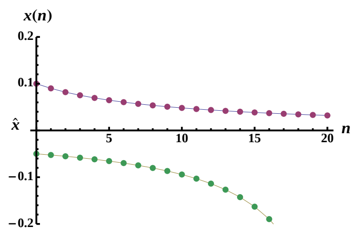


Figure 10.9: Dynamics of the model in (10.34) for two different values of $x(0)$ lying close to the equilibrium, but at different sides.

Exercises and assignments

1. Use Excel, R, or another device to calculate a few values of $x(n)$ for the quadratic model in (10.34). Choose starting values close to 0, on either side of the equilibrium.

2. Use Excel, R, or another device to examine the stability of the equilibrium at 0 for the model:

$$x(n+1) = x(n) \cdot (x(n) - 1) \tag{10.35}$$

10.4.3 Local dynamics of nonlinear models

For nonlinear systems, the solution equation can usually not be derived explicitly. Local properties of the dynamics of such models near equilibria can, however, be studied by approximating the nonlinear model by a linear model:

$$x(n+1) \approx a \cdot x(n) + b \tag{10.36}$$

where a and b are suitably chosen constants that depend on the model and the equilibrium values. In the following subsection the formal derivation of the approximation is given, and the expressions for a and b are derived. Here, we will just summarise the main results. It turns out that the proper choice for a is $f'(\hat{x})$, and, as a consequence, the following general results hold for nonlinear models:

- An equilibrium $\hat{x}$ is locally stable if $|f'(\hat{x})| < 1$.
- $\hat{x}$ is unstable if $|f'(\hat{x})| > 1$.
- Near the equilibrium, the solution $x(n)$ changes monotonically in time (either increasing or decreasing) if $f'(\hat{x}) > 0$.
- Near the equilibrium, the solution $x(n)$ oscillates in time (changes sign at every time step) if $f'(\hat{x}) < 0$.
- The equilibrium is a saddle point (attracting from one side and repelling from the other side, see Fig. 10.8) if $f'(\hat{x}) = 1$ and $f''(\hat{x}) \neq 0$.

As an example, consider the following model:

$$x(n+1) = \frac{1.5 \cdot x(n)}{1 + 3 \cdot x(n)} \tag{10.37}$$

The function $f(x)$ is in this case:

$$f(x) = \frac{1.5 \cdot x}{1 + 3 \cdot x} \tag{10.38}$$

With derivative:

See section 19.1.

$$f'(x) = \frac{1.5}{(1+3x)^2} \tag{10.39}$$

There are two equilibria: $\hat{x}_1 = 0$ and $\hat{x}_2 = 1/6$. Substituting these values in the derivative we find:

$$f'(0) = 1.5, f'\left(\frac{1}{6}\right) = \frac{2}{3} \tag{10.40}$$

and we can conclude that $\hat{x}_1$ is unstable and $\hat{x}_2$ is stable. Since the derivatives are positive, the local dynamics near each of the equilibria are monotonic.

Exercises and assignments

1. Consider the following model:

$$x(n+1) = \frac{2 \cdot x(n)}{1 + 5 \cdot x(n)} \tag{10.41}$$

   a. What is the function $f(x)$ that describes the relationship between $x(n+1)$ and $x(n)$ for this model?
   b. This model has two equilibria, what are their values?
   c. What is the derivative $f'(x)$?
   d. What are the values of $f'(\hat{x}_1)$ and $f'(\hat{x}_2)$?
   e. Which of the equilibria is stable, and which is unstable?
   f. Are the local dynamics near equilibria monotonic or not?

2. Consider the general Beverton-Holt model of Eq. (10.5).

a. What is the function $f(x)$ that describes the relationship between $x(n+1)$ and $x(n)$ for this model?

b. What is the derivative $f'(x)$?

c. For which parameter combinations is the equilibrium $\hat{x}_1 = 0$ stable?

d. Are the dynamics near the equilibrium monotonic or fluctuating?

e. Examine the stability of the other equilibrium point, $\hat{x}_2 = \frac{a-1}{b}$.

f. What can you say about the long term dynamics of this model if there are two equilibria?

3. The equilibria of the Ricker model of Eq, (10.6) were previously shown to be $\hat{x}_1 = 0$ and $\hat{x}_2 = \frac{\ln(a)}{b}$.

a. What is the function $f(x)$ that describes the relationship between $x(n+1)$ and $x(n)$ for this model?

b. What is the derivative $f'(x)$?

c. Use a first order approximation to determine for which parameter values the first equilibrium is stable, and for which it is unstable.

d. Use a first order approximation to determine for which parameter values the second equilibrium is stable or unstable.

e. What can you say about the type of dynamics around the second equilibrium: is it monotonic or fluctuating?

f. What is the value of the second derivative of $f(x)$ in the point $\hat{x}_1 = 0$, and what does this mean for the stability of this equilibrium? (Note: the state variable must be positive in this model!)

g. Compare the results of your local analysis with the dynamics for this model, that were shown in Fig. 10.3.

10.4.4 *Local dynamics: derivations and proofs*

This approximation is done by means of a first order Taylor approximation of the function $f(x)$ that specifies the relationship between $x(n)$ and $x(n+1)$, in the equilibrium point.

See section 19.4.1.

The first order Taylor approximation of $f(x)$ in the point $\hat{x}$ is:

$$f(x) \approx f(\hat{x}) + f'(\hat{x}) \cdot (x - \hat{x}) \tag{10.42}$$

Since $\hat{x}$ is an equilibrium point, $f(\hat{x}) = \hat{x}$ and thus:

$$f(x) \approx \hat{x} + f'(\hat{x}) \cdot (x - \hat{x}) \tag{10.43}$$

Therefore, the approximation of the model $x(n+1) = f(x(n))$ becomes

$$x(n+1) \approx \hat{x} + f'(\hat{x}) \cdot (x(n) - \hat{x}) \tag{10.44}$$

Rearranging and defining $z(n) = x(n) - \hat{x}$ gives:

$$\begin{aligned} x\left(n+1\right) - \hat{x} &\approx f'\left(\hat{x}\right) \cdot \left(x\left(n\right) - \hat{x}\right) \\ z\left(n+1\right) &= f'\left(\hat{x}\right) \cdot z\left(n\right) \end{aligned} \tag{10.45}$$

which is a linear model, with multiplication factor $f'\left(\hat{x}\right)$.

What happens if $f'\left(\hat{x}\right) = 1$?

If $f'\left(\hat{x}\right) = 1$ we may go a step further, and use a second order Taylor expansion, i.e.:

$$\begin{aligned} f\left(x\right) &\approx \hat{x} + f'\left(\hat{x}\right) \cdot \left(x - \hat{x}\right) + \frac{1}{2}f''\left(\hat{x}\right) \cdot \left(x - \hat{x}\right)^2 \\ x\left(n+1\right) &\approx \hat{x} + f'\left(\hat{x}\right) \cdot \left(x\left(n\right) - \hat{x}\right) + \frac{1}{2}f''\left(\hat{x}\right) \cdot \left(x\left(n\right) - \hat{x}\right)^2 \end{aligned} \tag{10.46}$$

Rearranging this, and replacing $x(n) - \hat{x}$ by $z(n)$ gives:

$$\begin{aligned} x\left(n+1\right) - \hat{x} &\approx f'\left(\hat{x}\right) \cdot \left(x\left(n\right) - \hat{x}\right) + \frac{1}{2}f''\left(\hat{x}\right) \cdot \left(x\left(n\right) - \hat{x}\right)^2 \\ z\left(n+1\right) &\approx f'\left(\hat{x}\right) \cdot z\left(n\right) + \frac{1}{2}f''\left(\hat{x}\right) \cdot z(n)^2 \end{aligned} \tag{10.47}$$

Since $f'(\hat{x}) = 1$:

$$\begin{aligned} z\left(n+1\right) &\approx z\left(n\right) + \frac{1}{2}f''\left(\hat{x}\right) \cdot z(n)^2 \\ &= z\left(n\right)\left(1 + \frac{1}{2}f''\left(\hat{x}\right) \cdot z\left(n\right)\right) \end{aligned} \tag{10.48}$$

This is a quadratic model, of the form:

$$x\left(n+1\right) = x\left(n\right) \cdot \left(1 + b \cdot x\left(n\right)\right) \tag{10.49}$$

with one equilibrium point at $\hat{x} = 0$. In section 10.4.2 we have studied a special case of this model with $b = -1$ (see Eq, (10.34)). There, the equilibrium 0 was found to be a saddle point. We will now show that this result is more general, namely that for any $b \neq 0$ the equilibrium 0 in the model of (10.49) is a saddle point.

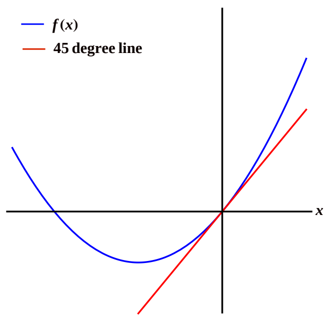


Figure 10.10: Graph of $f(x) = x \cdot (1 + b \cdot x)$ relative to the '$x = y$ line' when $b > 0$

First consider the case where $b > 0$ in that case the graph of $f(x) = x \cdot (1 + b \cdot x)$ is a u-shaped parabola (opening to the top). It has two roots: one at $x = 0$ and one at $x = -1/b$, which is negative since b is positive. The tangent line to the parabola in the point zero is the 45-degree line, where $y = x$, so all values of $f(x)$ are larger than x. This is illustrated in Fig. 10.10. Thus we can conclude that in this situation:

$$x\left(n+1\right) = x\left(n\right) \cdot \left(1 + b \cdot x\left(n\right)\right) > x\left(n\right) \tag{10.50}$$

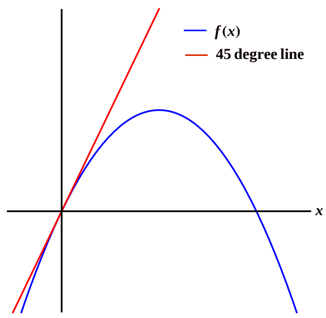


Figure 10.11: Graph of $f(x) = x \cdot (1 + b \cdot x)$ relative to the '$x = y$ line' when $b < 0$

Thus, the state variable always increases. This implies that if we choose a value to the left of $x = 0$ the state variable will move towards 0, so from this side the equilibrium is attractor. If we start on the right, however, the equilibrium is repelling. So $\hat{x} = 0$ is a saddle point.

Similarly, when $b < 0$, the graph of $f(x) = x \cdot (1 + b \cdot x)$ is a parabola opening to the bottom. In that case the root $x = -1/b$

is positive. Since, as before, the tangent line to the parabola in the point zero is the 45-degree line, all values of $f(x)$ are in this case smaller than x. This is illustrated in Fig. 10.11. Thus we can conclude that in this situation the state variable always decreases. Therefore the equilibrium at 0 is attracting from the right, and repelling from the left, and, again, $\hat{x} = 0$ is a saddle point.

We can conclude that for nonlinear models an equilibrium $\hat{x}$ with $f'(\hat{x}) = 1$ and $f''(\hat{x}) \neq 0$ is a saddle point.

If $f''(\hat{x}) = 0$ we need to look at yet higher order approximations to figure out the local stability properties of the equilibrium.

10.4.5 *Global analysis*

To examine global properties of nonlinear, univariate models, a graphical method may be used. This method is called the *cobweb method* for reasons that will become clear later on. It consists of plotting $x(n+1)$ against $x(n)$ and then using the graph to find subsequent values of the state variable. As an example, we consider the Ricker model (see Eq. (10.6)). As was shown in Fig. 10.3, this model may have many different types of dynamics. We will show here how this relates to its graph. In the Ricker model, the relationship between $x(n+1)$ and $x(n)$ is determined by the function:

$$f(x) = a \cdot x \cdot e^{-b \cdot x}, \text{ with } x \geq 0,\ a.0,\ b > 0 \qquad (10.51)$$

This function equals 0 at the point $x = 0$. It is positive for al $x > 0$, and it has a horizontal asymptote at $x = 0$ as x goes to infinity. By studying its derivative, you can see that it has a single maximum at $x = 1/b$. For this model, therefore, a graph of $x(n+1)$ versus $x(n)$ will have the general shape shown in Fig. 10.12. The graph also indicates how, for an arbitrary value of $x(0)$, the value of $x(1)$ can be found.

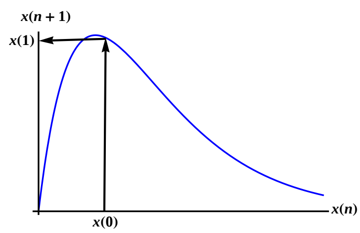


Figure 10.12: Relation between $x(n)$ and $x(n+1)$ for the Ricker model.

Once the value of $x(1)$ is known, the same method can be used to read the value of $x(2)$ from the graph. To do this, we need to project $x(1)$ on the horizontal axis, as shown in Fig. 10.13. Proceeding like this, we can find values of $x(3), x(4)$ etc. This is basically how the method works. There is one more 'trick' to it, though: plotting the 45°line in the graph provides a quick way to find the places of $x(1), x(2), \ldots$ on the horizontal axis. This is demonstrated in Fig. 10.14.

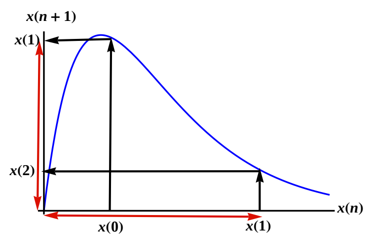


Figure 10.13: Finding the value of $x(2)$ from the graph.

Finally, it is no longer necessary to look for the values of $x(1), x(2), \ldots$ on the vertical axis, we can simply use the 45°line to find them. This is demonstrated in Fig. 10.15.

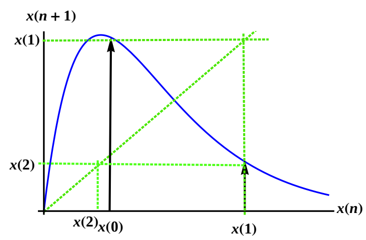


Figure 10.14: Finding the values of $x(1)$ and $x(2)$ from the graph, using the 45°line.

The cobweb method works as follows:

- Choose a value of $x(0)$,
- Find the value of x(1) of the curve, by drawing a vertical line.
- Draw a a horizontal line to the 45°line, this gives you the value of $x(1)$ on the horizontal axis.
- Draw a vertical line from the crossing point on the 45°line to the curve. This gives you the value of $x(2)$

- Draw a a horizontal line to the 45°line, this gives you the value of $x(2)$ on the horizontal axis
- Repeat...

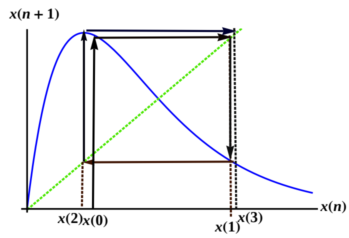


Figure 10.15: Illustration of the cobweb method.

If you proceed with this method in Fig. 10.15, you can see where it gets its name. The method is not meant to predict a large sequence of $x(n)$ values, but rather to find out the regions of attraction of locally stable equilibria, or to find out what happens in the neighbourhood of unstable ones. Note that equilibria show up in the graph as points where the 45°line crosses the curve, since these are the points where $x(n+1) = x(n)$. In the situation shown in the graph, there are two equilibria.

The graphical method, together with the local analysis will give you an idea of the dynamics of the model. In the situation of Fig. 10.15, there are two unstable equilibria. The graphical analysis shows that the state variable moves away from the first equilibrium point (you may check by starting with values of $x(0)$ close to the point 0), but starts oscillating around the second equilibrium. Additional, numerical analysis (using computer software) may then be used to find out whether there are stable cycles or chaos. The way to find this out is to start with initial values close to the second, unstable, equilibrium, and studying the long-term dynamics.

Exercises and assignments

Use the cobweb method in combination with local analysis and numerical methods to find out the dynamics of the following models for several combinations of parameter values (choose your own values):

1. $x(n+1) = \frac{a \cdot x(n)}{1+b \cdot x(n)} - h \cdot x$, with $x \geq 0$ and $a, b, h > 0$
2. $x(n+1) = a \cdot x(n) \cdot e^{-b \cdot x(n)} - h \cdot x$, with $x \geq 0$ and $a, b, h > 0$

11
Deterministic, discrete time, multivariate models

11.1 Linear, two-dimensional models

11.1.1 Notation

Multivariate models contain several state variables. We will use subscripts to distinguish those. For instance, in a two dimensional model there are two state variables, x_1 and x_2. Such a model is specified by two recurrence relationships: one for each of the state variables. In a linear model, these relationships are linear, for example:

$$\begin{aligned} x_1(n+1) &= 2 \cdot x_1(n) - 3 \cdot x_2(n) \\ x_2(n+1) &= 3 \cdot x_1(n) + 2 \cdot x_2(n) \end{aligned} \tag{11.1}$$

When the values of both state variables at time n are known, these two equations may be used to compute their values at time $n+1$. For instance, if $x_1(n) = 1$, and $x_2(n) = 3$ then

$$\begin{aligned} x_1(n+1) &= 2 \cdot 1 - 3 \cdot 3 = -7 \\ x_2(n+1) &= 3 \cdot 1 + 2 \cdot 3 = 9 \end{aligned} \tag{11.2}$$

The system of equations in (11.1) may also be written in matrix notation. Define a vector $\mathbf{x}(\mathbf{n})$ with the state variable values at time n as elements, and a matrix $\mathbf{A}$ as follows:

For an introduction to matrices and vectors see chapter 21.

$$\mathbf{x}(\mathbf{n}) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \tag{11.3}$$

Then the system of equations in (11.1) corresponds to:

$$\mathbf{x}(\mathbf{n}+1) = \mathbf{A} \cdot \mathbf{x}(\mathbf{n}) \tag{11.4}$$

More generally, a discrete time 2-dimensional linear model has the form:

$$\begin{aligned} x_1(n+1) &= a \cdot x_1(n) + b \cdot x_2(n) \\ x_2(n+1) &= c \cdot x_1(n) + d \cdot x_2(n) \end{aligned} \tag{11.5}$$

and the matrix $\mathbf{A}$ corresponds to:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{11.6}$$

Exercises and assignments

1. What are the values of the two state variables at time $n+1$ in the model of Eq. (11.3), if $x_1(n) = 2$, and $x_2(n) = 1$?

2. What is the matrix $\mathbf{A}$ in each of the following cases?

   a.
   $$\begin{aligned} x_1(n+1) &= 5 \cdot x_1(n) + x_2(n) \\ x_2(n+1) &= 2 \cdot x_1(n) - 2 \cdot x_2(n) \end{aligned}$$

   b.
   $$\begin{aligned} x_1(n+1) &= x_1(n) + x_2(n) \\ x_2(n+1) &= x_1(n) - x_2(n) \end{aligned}$$

   c.
   $$\begin{aligned} x_1(n+1) &= x_2(n) \\ x_2(n+1) &= x_1(n) - x_2(n) \end{aligned}$$

3. For each of the models in the previous question, give the values of the state variables at times 1 and 2, if the initial condition is $x_1(0) = 1, x_2(0) = 1$.

11.1.2 *The equilibrium*

From (11.4) we find the equilibrium equation for a multidimensional linear model:

$$\hat{\mathbf{x}} = \mathbf{A} \cdot \hat{\mathbf{x}} \tag{11.7}$$

where $\hat{\mathbf{x}}$ is a column vector of equilibrium values. Depending on the matrix $\mathbf{A}$ this equation may have a single solution, $\hat{\mathbf{x}} = \mathbf{0}$, or an infinite number of solutions. We will only consider the first type of situations here.

This implies assuming that the matrix $\mathbf{A}$ has full rank, which, for most applications, will be the case.

For two-dimensional models, (11.7) corresponds to a system of two linear equations. For instance, for the example in Eq. (11.1):

$$\begin{aligned} \hat{x}_1 &= 2 \cdot \hat{x}_1 - 3 \cdot \hat{x}_2 \\ \hat{x}_2 &= 3 \cdot \hat{x}_1 + 2 \cdot \hat{x}_2 \end{aligned} \tag{11.8}$$

Note that $\hat{x}_1$ and $\hat{x}_2$ denote the coordinates of a *single* equilibrium point $\hat{\mathbf{x}}$.

The first equation implies $\hat{x}_1 = 3 \cdot \hat{x}_2$, and the second $\hat{x}_1 = -1/3 \cdot \hat{x}_2$, the two equations are satisfied only if $3 \cdot \hat{x}_2 = -1/3 \cdot \hat{x}_2$, which means that $\hat{x}_2$ must indeed be zero. Since $\hat{x}_1 = 3 \cdot \hat{x}_2$ this implies that $\hat{x}_1 = 0$ too. This demonstrates that $(0, 0)$ is indeed the only equilibrium point.

Exercises and assignments

1. Write down the equilibrium equations for the following model, and demonstrate that $(0,0)$ is the only outcome:

$$x_1(n+1) = 5 \cdot x_1(n) + x_2(n)$$
$$x_2(n+1) = 2 \cdot x_1(n) - 2 \cdot x_2(n)$$

2. Consider the more general model representation in (11.5).

   a. What are the equilibrium equations?

   b. Show that, when $a = 1$ and all other parameters are non-zero the only equilibrium point is $(0,0)$.

   c. What are the equilibria when $a = 1$, $c = 0$, and b and d are non-zero?

11.1.3 The solution equation

Just as with univariate models, a system of recurrence equations specifies a unique sequence of values of the state variables as soon as an initial condition is given. In the multivariate case, an initial condition corresponds to an initial value for each of the state variables. For instance, the model of (11.1) in combination with the initial condition $x_1(0) = 1$, $x_2(0) = 2$ leads to $x_1(1) = -4$, $x_2(1) = 7$. Filling in these values in the recurrence relation gives $x_1(2) = -29$, $x_2(2) = 2$, and so on.

In matrix notation, the initial condition is a vector $\mathbf{x}(\mathbf{0})$, and, using the specification in (11.4) we get the solution:

$$\begin{aligned}
\mathbf{x}(1) &= \mathbf{A} \cdot \mathbf{x}(0) \\
\mathbf{x}(2) &= \mathbf{A} \cdot \mathbf{x}(1) = \mathbf{A}^2 \cdot \mathbf{x}(0) \\
&\vdots \\
\mathbf{x}(\mathbf{n}) &= \mathbf{A}^n \cdot \mathbf{x}(\mathbf{0})
\end{aligned} \tag{11.9}$$

which looks quite similar to the solution of the one-dimensional linear model (see (10.15)). In this case, however, the symbol $\mathbf{A}^\mathbf{n}$ denotes a matrix product. The properties of this matrix product determine the stability of the equilibrium, and the type of dynamics of the state variables.

It turns out that in a 2-dimensional system, the dynamics can be inferred directly from two properties of the matrix $\mathbf{A}$, namely its *trace* and its *determinant*.

See section 21.3 for their definition and some exercises.

Exercises and assignments

1. For each of the following models, give the trace and determinant of the matrix $\mathbf{A}$:

   a.

$$x_1(n+1) = \frac{1}{2} \cdot x_1(n) - x_2(n)$$
$$x_2(n+1) = x_1(n) + 3 \cdot x_2(n)$$

b.

$$x_1(n+1) = -x_2(n)$$
$$x_2(n+1) = 2x_1(n) + 5 \cdot x_2(n)$$

c.

$$x_1(n+1) = -2 \cdot x_1(n) + x_2(n)$$
$$x_2(n+1) = x_1(n) - x_2(n)$$

d.

$$x_1(n+1) = x_2(n)$$
$$x_2(n+1) = x_1(n)$$

11.1.4 *Dynamics*

The solutions of the two-dimensional linear model can have several forms: they move either towards or away from the equilibrium point, where all state variables are zero, and they can do so monotonically or with fluctuations. There may also be situations where the solutions stably fluctuate around the origin. What happens depends on the matrix **A**. As mentioned in the previous section, for the two dimensional case, this is completely determined by the combination of values of its trace, T and its determinant, D. The main characteristics are:

- The equilibrium is stable if the following three inequalities are all satisfied: $D < 1$ and $D > T - 1$ and $D > -T - 1$
- The values of the state variables fluctuate when $D > \frac{1}{4}T^2$
- When $D < \frac{1}{4}T^2$, the values of $x_1(n)$ and $x_2(n)$ either change monotonically, or 'zig-zag' (see e.g. Figs.11.2 and 11.8).

Figure 11.1 gives a summary of all the different types of dynamics for the different combinations of values of D and T.

We will give an overview of the different types of equilibria and the corresponding dynamics below, with some examples. To illustrate the dynamics, we will show two types of graphs. One graph is the usual one, with the values of the state variables $x_1(n)$ and $x_2(n)$ versus time, n. The other graph shows $x_1(n)$ on the horizontal axis, and the corresponding value of $x_2(n)$ on the vertical axis. This type of plot is called a *phase plot*. Phase plots provide additional insight in the joined dynamics of the state variables, for instance to detect whether there are stable cycles.

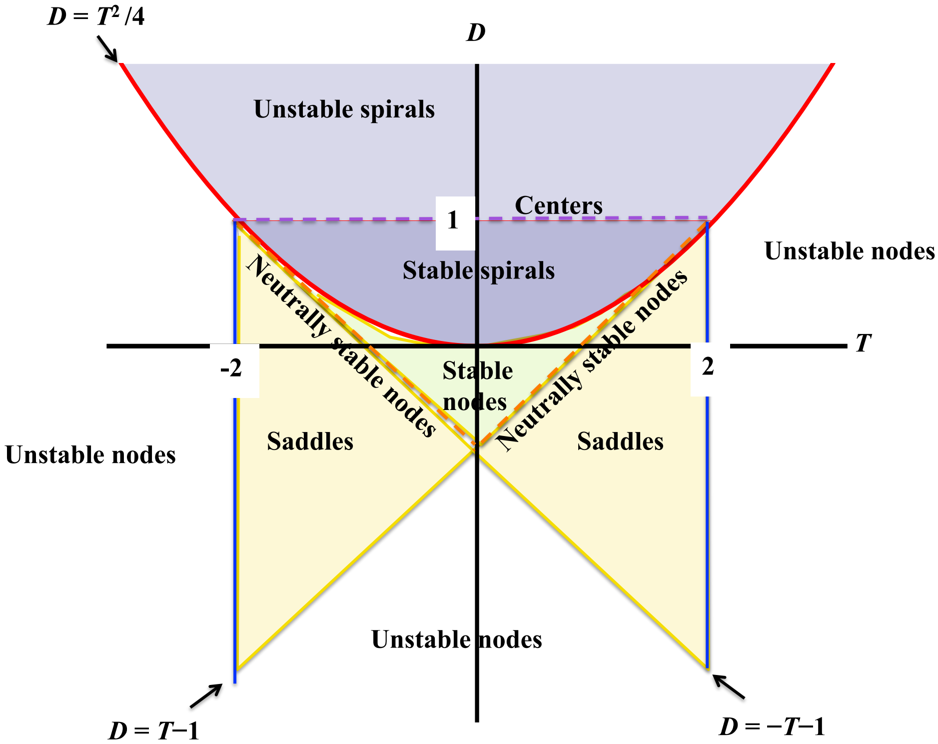


Figure 11.1: Overview of the different types of dynamics of the linear two-dimensional model, for different combinations of D and T. See text below for further explanation.

Stable equilibria:

- ***Stable node***: $\max(T-1, -T-1) < D \leq \frac{1}{4}T^2$
  Example:

$$\mathbf{A} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{4} \end{pmatrix} \tag{11.10}$$

For this example, $T = 1/4$, so $-T-1 = -5/4$, and $T-1 = -3/4$. Thus, $\max(T-1, -T-1) = -3/4$. Furthermore $\frac{1}{4}T^2 = 1/64$, and $D = -1/2$. Figure11.2 shows an example of the typical dynamics of the solution. The bottom figure shows the phase plot. Here the solution moves from right to left, since the values of both state variables decrease in time.

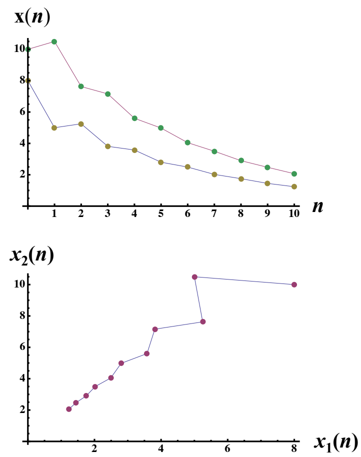


Figure 11.2: Dynamics (top) and phase plot (bottom), for the linear model with $\mathbf{A}$ as in Eq. (11.10), and $x_1(0) = 8, x_2(0) = 10$. Top: $x_1(n)$ brown and $x_2(n)$ green.

- ***Stable spiral*** : $\frac{1}{4}T^2 < D < 1$
  Example:

$$\mathbf{A} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 1 & 1 \end{pmatrix} \tag{11.11}$$

  For this example, $T = 1$, so $\frac{1}{4}T^2 = 1/4$, and $D = 1/2$. The equilibrium is stable, and the state variables fluctuate. Because the equilibrium is stable, the fluctuations decrease as the solution approaches zero. This is demonstrated in Fig. 11.3. Note that in the phase plot the solution spirals into zero.

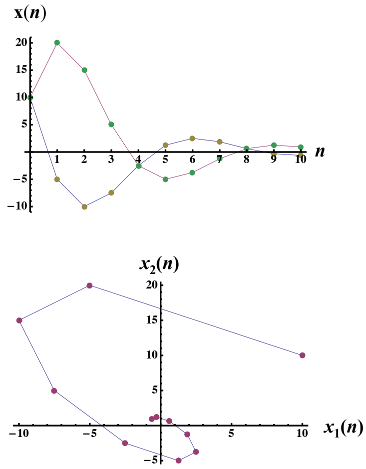


Figure 11.3: Dynamics (top) and phase plot (bottom), for the linear model with **A** as in Eq. (11.11), and $x_1(0) = 10, x_2(0) = 10$. Top: $x_1(n)$ brown and $x_2(n)$ green.

Neutrally stable equilibria

- ***Neutrally stable node*** : $0 < T < 2$ and $D = T - 1$
  Example:

$$\mathbf{A} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{3}{2} \end{pmatrix} \tag{11.12}$$

  For this example, $T = 3/2$, so $T - 1 = 1/2$, and $D = 1/2$. The solution will converge to a point at a fixed distance from the origin. Thus, the origin is neither attracting nor repelling. This is illustrated in Fig. 11.4. In the phase plot the solution moves from left to right. Note that the coordinates x_1, x_2 of the final state depend on the initial condition. For a different combination of $x_1(0)$ and $x_2(0)$ the final outcome would be different.

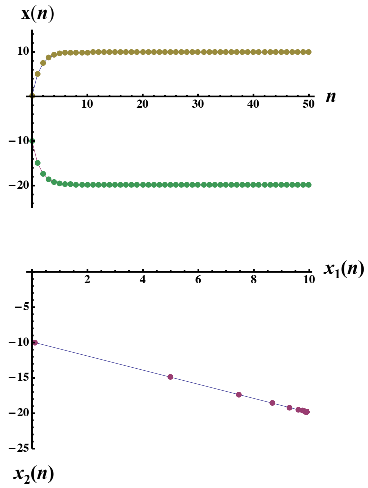


Figure 11.4: Dynamics (top) and phase plot (bottom),for the linear model with **A** as in Eq. (11.12), and $x_1(0) = 0.1, x_2(0) = -10$. Top: $x_1(n)$ brown and $x_2(n)$ green.

- ***Neutrally stable node*** : $-2 < T < 0$ and $D = -T - 1$
  In this situation the solutions alternate between two values that lie at different sides of the equilibrium. For example:

$$\mathbf{A} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \tag{11.13}$$

  In this example, $T = -1/2$, so $-T - 1 = -1/2$, and $D = -1/2$.The solution converges to a stable alternation between two points equidistant from, but on opposite sides of the origin. The position of these points depend on the initial condition. Fig. 11.5 shows an example. In the phase plot the solution starts on the right and moves to the left.

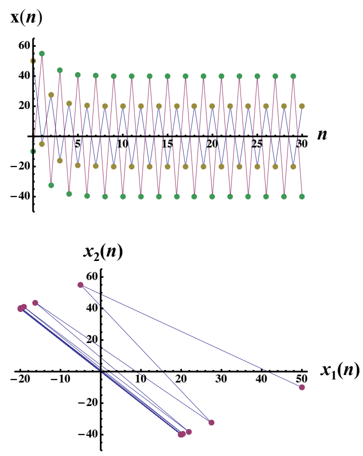


Figure 11.5: Dynamics (top) and phase plot (bottom), for the linear model with **A** as in Eq. (11.13), and $x_1(0) = 50, x_2(0) = -10$. Top: $x_1(n)$ brown and $x_2(n)$ green.

- ***Center***: $D = 1$ and $-2 < T < 2$
Example:
$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \tag{11.14}$$
For this example, $T = 1$, and $D = 1$.This equilibrium is *neutrally stable*. For any initial condition, the solution will neither move away nor move towards the equilibrium. Instead, it will keep following a cycle around the equilibrium. This is illustrated in Fig. 11.6. Note that the initial conditions determine the cycle. Choosing other initial values for the state variable will lead to a different cycle. For this example, the solution proceeds clockwise in the phase plot.

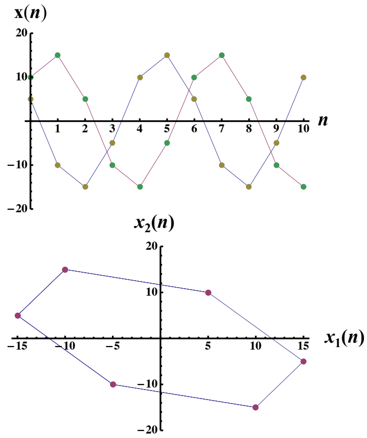


Figure 11.6: Dynamics (top) and phase plot (bottom), for the linear model with **A** as in Eq. (11.14), and $x_1(0) = 5, x_2(0) = 10$. Top: $x_1(n)$ brown and $x_2(n)$ green.

Unstable equilibria

- ***Saddle point***: either: $-2 < T < 0$ and $T - 1 < D < -T - 1$ or: $0 < T < 2$ and $-T - 1 < D < T - 1$
Example:
$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} \tag{11.15}$$
In this example, $T = 3$, so $-T - 1 = -4$, $T - 1 = 2$, and $D = 1$, so $-T - 1 < D < T - 1$. The equilibrium is attracting in one direction, and repelling in another direction.The solution may move initially in the direction of 0 for some initial conditions. This is demonstrated in Fig. 11.7. Note, however, that in the long run the state variable moves away from the equilibrium point. In the phase plot, the solution starts in the upper left corner, then moves towards the origin, and but verges away from there once it gets near.

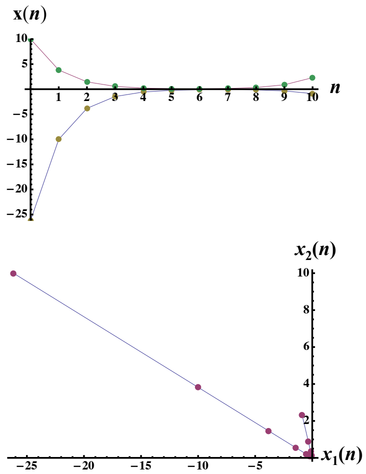


Figure 11.7: Dynamics (top) and phase plot (bottom), for the linear model with **A** as in Eq. (11.15), and $x_1(0) = -26.18, x_2(0) = 10$. Top: $x_1(n)$ brown and $x_2(n)$ green.

- ***Unstable node*** : $D < \min(T - 1, -T - 1)$, or $1 < \frac{1}{4}T^2 < D$
Example:
$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \tag{11.16}$$
In this example, $T = 0$, so $-T - 1 = -1$, $T - 1 = -1$, $\frac{1}{4}T^2 = 0$, and $D = -4$. The equilibrium is unstable, and repelling in all directions. Thus, no matter how close the initial values of the state variables lie to 0, the solution will move away from this equilibrium point. In the chosen example, this happens in a 'zig-zag' fashion. In other situations the divergence from the equilibrium may occur more smoothly. The dynamics for this example are illustrated in Fig. 11.8. Note that the solution moves from left to right in the phase plot.

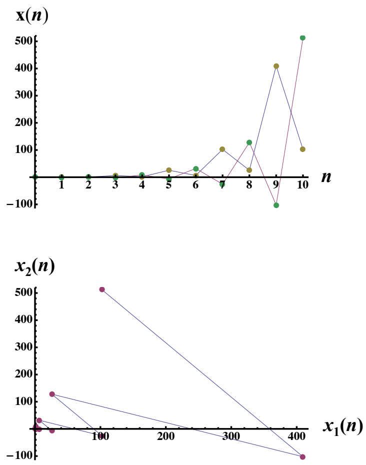


Figure 11.8: DynamicsDynamics (top) and phase plot (bottom), for the linear model with **A** as in Eq. (11.16), and $x_1(0) = 0.1, x_2(0) = 0.5$. Top: $x_1(n)$ brown and $x_2(n)$ green.

- ***Unstable spiral*** :$D > \frac{1}{4}T^2$, and $D > 1$
  Example:
  $$\mathbf{A} = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} \tag{11.17}$$
  In this example $T = 1$ and $D = 2$. The equilibrium is unstable, and, since $D > \frac{1}{4}T^2$, the solutions fluctuate with an increasing amplitude. This is demonstrated in Fig. 11.9. In the phase plot the solution spirals away from zero.

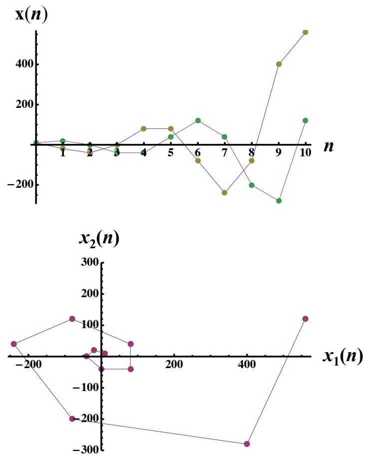


Figure 11.9: Dynamics (top) and phase plot (bottom), for the linear model with **A** as in Eq. (11.11), and $x_1(0) = 10, x_2(0) = 10$. Top: $x_1(n)$ brown and $x_2(n)$ green.

Exercises and assignments

1. Examine the dynamics of the solutions for the different examples numerically, using different initial conditions than those in the figures.

2. Determine the type of equilibrium for the linear models corresponding to the following matrices **A**:

   a.
   $$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$
   b.
   $$\begin{pmatrix} -1 & 1 \\ 1/2 & 0 \end{pmatrix}$$
   c.
   $$\begin{pmatrix} 1/2 & 1 \\ -3 & 0 \end{pmatrix}$$
   d.
   $$\begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}$$

3. Determine for which values of the parameter α the equilibrium of the linear models corresponding to the following matrices **A** is stable:

   a.
   $$\begin{pmatrix} \alpha & 2 \\ 1 & 0 \end{pmatrix}$$
   b.
   $$\begin{pmatrix} 1 & \alpha \\ 1 & 0 \end{pmatrix}$$
   c.
   $$\begin{pmatrix} 0 & -\alpha \\ 1 & 1 \end{pmatrix}$$

4. Construct a matrix **A** with $D = T - 1$ or $D = -T - 1$, and examine the dynamics of the corresponding linear model.

11.1.5 The general linear model

In this subsection we briefly consider the more general linear model:

$$\mathbf{x}\left(\mathbf{n}+\mathbf{1}\right)=\mathbf{A}\cdot\mathbf{x}\left(\mathbf{n}\right)+\mathbf{b} \tag{11.18}$$

where $\mathbf{b}$ is a column vector of two constants.

An example of such a model is:

$$\begin{aligned} x_1(n+1) &= 2\cdot x_1(n)+x_2(n)-3 \\ x_2(n+1) &= -x_1(n)+x_2(n)+1 \end{aligned} \tag{11.19}$$

In matrix notation:

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\cdot\begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}+\begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{11.20}$$

We will show that this type of models can be transformed to the previously considered ones, and thus very similar conclusions can be drawn concerning their dynamics.

The equilibrium equation for this model is:

$$\hat{\mathbf{x}}=\mathbf{A}\cdot\hat{\mathbf{x}}+\mathbf{b} \tag{11.21}$$

This gives two equations with two unknowns, $\hat{x}_1$ and $\hat{x}_2$. For instance, for (11.20) we get:

$$\begin{aligned} \hat{x}_1 &= 2\cdot\hat{x}_1+\hat{x}_2-3 \\ \hat{x}_2 &= -\hat{x}_1+\hat{x}_2+1 \end{aligned} \tag{11.22}$$

which can be solved to give $\hat{x}_1=1,\ \hat{x}_2=2$.

Subtraction of $\hat{\mathbf{x}}$ on both sides in (11.18) and rearranging the result gives:

$$\begin{aligned} \mathbf{x}\left(\mathbf{n}+\mathbf{1}\right)-\hat{\mathbf{x}} &= \mathbf{A}\cdot\mathbf{x}\left(\mathbf{n}\right)+\mathbf{b}-\hat{\mathbf{x}} \\ &= \mathbf{A}\cdot\mathbf{x}\left(\mathbf{n}\right)+\mathbf{b}-\left(\mathbf{A}\cdot\hat{\mathbf{x}}+\mathbf{b}\right) \\ &= \mathbf{A}\cdot\left(\mathbf{x}\left(\mathbf{n}\right)-\hat{\mathbf{x}}\right) \end{aligned} \tag{11.23}$$

where we have used (11.21) in the second step. Thus, if we define

$$\mathbf{z}\left(\mathbf{n}\right)=\mathbf{x}\left(\mathbf{n}\right)-\hat{\mathbf{x}} \tag{11.24}$$

the recurrence relation for $\mathbf{z}(\mathbf{n})$ is:

$$\mathbf{z}\left(\mathbf{n}+1\right)=\mathbf{A}\cdot\mathbf{z}\left(\mathbf{n}\right) \tag{11.25}$$

Since $\mathbf{z}(\mathbf{n})$ is the distance to the equilibrium point all results of the previous mode concerning the dynamics around the equilibrium can be directly applied to models of the form in (11.21). Thus, all that matters is the matrix $\mathbf{A}$ and its trace and determinant. The only effect of $\mathbf{b}$ is to determine the location of the equilibrium point.

Exercises and assignments

1. Compute the trace and determinant of the matrix $\mathbf{A}$ for the model in (11.20). What can you say concerning the model dynamics? Check your predictions numerically, using several different initial conditions.

11.2 *Nonlinear two-dimensional models*

A nonlinear, two-dimensional, discrete time model is determined by two nonlinear recurrence equations, for instance:

$$\begin{aligned} x_1(n+1) &= 2 \cdot x_1(n) - \frac{1}{2} x_1(n) \cdot x_2(n) \\ x_2(n+1) &= \frac{1}{3} x_1(n) \cdot x_2(n) - x_2(n) \end{aligned} \tag{11.26}$$

A more general notation for such a model is:

These equations are nonlinear because they contain the product of $x_1(n)$ and $x_2(n)$.

$$\begin{aligned} x_1(n+1) &= f_1\left(x_1(n), x_2(n)\right) \\ x_2(n+1) &= f_2\left(x_1(n), x_2(n)\right) \end{aligned} \tag{11.27}$$

In the example of (11.26) the functions f_1 and f_2 are defined as:

$$\begin{aligned} f_1\left(x_1, x_2\right) &= 2x_1 - \frac{1}{2} x_1 \cdot x_2 \\ f_2\left(x_1, x_2\right) &= \frac{1}{3} x_1 \cdot x_2 - x_2 \end{aligned} \tag{11.28}$$

Multivariate models may also be represented in matrix notation, as:

$$\mathbf{x}\left(\mathbf{n}+\mathbf{1}\right) = \mathbf{F}\left(\mathbf{x}\left(\mathbf{n}\right)\right) \tag{11.29}$$

where $\mathbf{x}\left(\mathbf{n}\right)$ denotes a vector of the values of the state variables at time n, and $\mathbf{F}\left(\mathbf{x}\right)$ is a vector valued function:

$$\mathbf{F}\left(\mathbf{x}\right) = \begin{pmatrix} f_1\left(x_1, x_2\right) \\ f_2\left(x_1, x_2\right) \end{pmatrix} \tag{11.30}$$

The advantage of this notation is that it clearly shows the analogy with univariate models.

Just as with one-dimensional systems, the dynamics of the state variables of multi-dimensional systems may be studied by means of equilibria and their local properties, based on linear approximations.

Exercises and assignments

1. What are the values of the state variables at times 1 and 2 for the model in (11.26), with initial condition $x_1(0) = 1$ and $x_2(0) = 1$?

2. What are the functions $f_1\left(x_1, x_2\right)$ and $f_2\left(x_1, x_2\right)$ in the following model?
$$\begin{aligned} x_1(n+1) &= 2x_1(n) \cdot \left(1 - x_2(n)\right) \\ x_2(n+1) &= x_1(n) \end{aligned}$$

3. What are the functions $f_1\left(x_1, x_2\right)$ and $f_2\left(x_1, x_2\right)$ in the following model?
$$\begin{aligned} x_1(n+1) &= x_1(n) \cdot \left(1 - \frac{x_1(n)}{K}\right) - \alpha \cdot x_1(n) \cdot x_2(n) \\ x_2(n+1) &= \lambda \cdot \alpha \cdot x_1(n) \cdot x_2(n) - \mu \cdot x_2(n) \end{aligned}$$

11.2.1 *Equilibria of nonlinear models*

As in the univariate case, equilibria are points in which no change occurs. In the multivariate case this means that all the state variables must maintain the same value. Thus, if $\hat{\mathbf{x}}$ is an (vector valued) equilibrium point, then, if $\mathbf{x}(\mathbf{n}) = \hat{\mathbf{x}}$, it follows that $\mathbf{x}(\mathbf{n}+\mathbf{1}) = \hat{\mathbf{x}}$. This leads to the equilibrium equation:

$$\hat{\mathbf{x}} = \mathbf{F}(\hat{\mathbf{x}}) \tag{11.31}$$

which is completely analogous to the univariate case. Thus, the equilibria are found by solving the system of equations:

$$\begin{aligned} \hat{x}_1 &= f_1(\hat{x}_1, \hat{x}_2) \\ \hat{x}_2 &= f_2(\hat{x}_1, \hat{x}_2) \end{aligned} \tag{11.32}$$

For the example in (11.26) these are:

$$\begin{aligned} \hat{x}_1 &= 2 \cdot \hat{x}_1 - \frac{1}{2}\hat{x}_1 \cdot \hat{x}_2 \\ \hat{x}_2 &= \frac{1}{3}\hat{x}_1 \cdot \hat{x}_2 - \hat{x}_2 \end{aligned} \tag{11.33}$$

The first equilibrium equation gives:

$$\hat{x}_1 = 2 \cdot \hat{x}_1 - \frac{1}{2}\hat{x}_1 \cdot \hat{x}_2 \Rightarrow \hat{x}_1 = 0, \text{ or } 2 - \frac{1}{2}\hat{x}_2 = 1 \Rightarrow \hat{x}_2 = 2 \tag{11.34}$$

Substituting $\hat{x}_1 = 0$ in the second equation gives $\hat{x}_2 = -\hat{x}_2$, which has solution $\hat{x}_2 = 0$. Thus, one of the equilibria is the point $(0, 0)$. Substituting $\hat{x}_2 = 2$ in the second equation gives:

$$2 = \frac{2}{3}\hat{x}_1 - 2 \Rightarrow \frac{2}{3}\hat{x}_1 = 4 \Rightarrow \hat{x}_1 = \frac{12}{2} = 6. \tag{11.35}$$

Thus, the other equilibrium is $(6, 2)$.

Exercises and assignments

1. Show that $(0, 2)$ and $(6, 0)$ are not equilibrium points of the model in (11.26).

2. Consider the following model:

$$\begin{aligned} x_1(n+1) &= x_1(n) \cdot x_2(n) - 2 \cdot x_1(n) \\ x_2(n+1) &= x_2(n) \cdot (2 - x_2(n)) - 3 \cdot x_1(n) \cdot x_2(n) \end{aligned}$$

   a. Give the equilibrium equations for this model.

   b. Give the coordinates of the three equilibrium points for this model.

11.2.2 *Local properties*

As in the univariate case, the dynamics in the vicinity of equilibria for nonlinear multivariate models can be studied by approximating the model by a linear system. The multivariate analogue to the first order approximation in (10.36) is:

$$\mathbf{x}\left(\mathbf{n}+\mathbf{1}\right) \approx \mathbf{A} \cdot \mathbf{x}\left(\mathbf{n}\right) + \mathbf{b} \tag{11.36}$$

where $\mathbf{A}$ is a matrix, and $\mathbf{b}$ a column vector. Both depend on the equilibrium value and the shape of the nonlinear model. The approximation is a linear system (see e.g. (11.25)). Its dynamics depend on the properties of $\mathbf{A}$ (cf. section 11.1.4).

It turns out that the matrix $\mathbf{A}$ corresponds to the *Jacobian matrix*, or *Jacobian*, of the vector valued function $\mathbf{F}\left(\mathbf{x}\right)$, evaluated at the equilibrium point. The Jacobian, here denoted by $\mathbf{J}(\mathbf{x})$, is a square matrix, with elements that are equal to the partial derivatives of the functions in $\mathbf{F}$. For the two-dimensional case:

Partial derivatives are explained in section 19.3.

$$\mathbf{J}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \tag{11.37}$$

For instance, consider the example of (11.26). To approximate this model, we need to calculate the partial derivatives of the funcions in Eq. (11.28)). These are:

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= \frac{\partial}{\partial x_1}\left(2x_1 - \frac{1}{2}x_1 \cdot x_2\right) = 2 - \frac{1}{2}x_2 \\
\frac{\partial f_1}{\partial x_2} &= \frac{\partial}{\partial x_2}\left(2x_1 - \frac{1}{2}x_1 \cdot x_2\right) = -\frac{1}{2}x_1 \\
\frac{\partial f_2}{\partial x_1} &= \frac{\partial}{\partial x_1}\left(\frac{1}{3}x_1 \cdot x_2 - x_2\right) = \frac{1}{3}x_2 \\
\frac{\partial f_2}{\partial x2} &= \frac{\partial}{\partial x_2}\left(\frac{1}{3}x_1 \cdot x_2 - x_2\right) = \frac{1}{3}x_1 - 1
\end{aligned} \tag{11.38}$$

So, for this model the Jacobian matrix is:

$$\mathbf{J}\left(\mathbf{x}\right) = \begin{pmatrix} 2 - \frac{1}{2}x_2 & -\frac{1}{2}x_1 \\ \frac{1}{3}x_2 & \frac{1}{3}x_1 - 1 \end{pmatrix} \tag{11.39}$$

For the approximation in a specific equilibrium point, the coordinates of that point are substituted in the Jacobian. For instance, we derived before that the model in (11.26) has two equilibrium points: $(0,0)$ and $(3,4)$ (see (11.33)). To study local dynamics near the first equilibrium point, substituting the value 0 for both state variables in the Jacobian gives:

$$\mathbf{J}\left(\hat{\mathbf{x}}\right) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \tag{11.40}$$

The trace and determinant of this matrix are $T = -1, D = -2$, so $T - 1 = -2, -T - 1 = 0$, $D = T - 1, D < -T - 1$. The equilibrium $(0,0)$ is an unstable node (see section 11.1.4).

Exercises and assignments

1. What is the Jacobian matrix for the model in (11.26), evaluated at the other equilibrium point, $(3,4)$, and what type of equilibrium is this?

2. Consider the following model:

$$\begin{aligned} x_1(n+1) &= -x_1(n) + \frac{1}{2}x_1(n)\cdot x_2(n) \\ x_2(n+1) &= -\frac{1}{2}x_1(n)\cdot x_2(n) + 5x_2(n) \end{aligned}$$

   a. What are the functions $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ for this model?
   b. Show that $(0,0)$ is an equilibrium point of this model.
   c. Find the coordinates of the second equilibrium point for this model.
   d. Determine the Jacobian matrix for this model.
   e. Use a first order approximation to predict the local dynamics of the state variable near the equilibrium (0,0). What type of equilibrium is this?
   f. Do the same for the other equilibrium point.

3. Consider the following model:

$$\begin{aligned} x_1(n+1) &= \frac{1}{2}x_1(n)\cdot x_2(n) - 2\cdot x_1(n) \\ x_2(n+1) &= x_2(n)\cdot\left(1-\frac{x_2(n)}{3}\right) - x_1(n)\cdot x_2(n) \end{aligned}$$

   a. Show that $(0,0)$ is an equilibrium point of this model.
   b. Determine the Jacobian matrix for this model.
   c. Use a first order approximation to predict the local dynamics of the state variable near the equilibrium $(0,0)$. What type of equilibrium is this?

11.3 *Higher-dimensional models*

The analysis of models with three or more state variables is similar to that described for two-dimensional models in the previous sections. Solving the equilibrium equation (11.31) may be much harder, especially if the model's dimension is large. In many cases one must resort to numerical (computer) methods for finding the equilibria. The same holds for analysis of local dynamics near equilibria. As in two-dimensional models, the stability of an equilibrium point is determined by the properties of the Jacobian matrix, evaluated at that point. As opposed to the 2-dimensional case, though, there is no simple connection with quantities that can easily be computed, such as the trace and the determinant. Only in rare cases it is possible to examine this algebraically when there are more than three state variables. Therefore, numerical methods, available in computer packages, are used.

12

Deterministic, continuous time, univariate models

In this chapter we consider situations where a variable may change at any moment in time. Just as with discrete time models, such a variable is called a *state variable*. It is assumed that, given its value at a specific moment, we can fully predict its future values. Hence, these are deterministic models.

12.1 What do they look like?

The type of models that we consider here are described by *differential equations*, such as, for instance:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2 \cdot x \tag{12.1}$$

where x is the state variable and t is time. It is important to note that, although it is usually not specified explicitly, x is a function of time. The differential quotient on the left-hand side corresponds to the derivative of this function. Thus, another way to write(12.1) is:

Important! A differential equation of the form in (12.1) defines the relationship between the the derivative $x'(t)$ and $x(t)$.

$$x'(t) = 2 \cdot x(t) \tag{12.2}$$

Yet another notation that you might encounter in the literature is:

$$\dot{x} = 2 \cdot x \tag{12.3}$$

Throughout this book we will use the notation in (12.1), which is the most commonly used notation.

A differential equation describes the relationship between the value of x at time t and the instantaneous change in x at that moment. So, for instance, when $x(t) = 2$, its instantaneous change is 4, and when $x(t) = 4$, its instantaneous change is 8. In a graph of $x(t)$ versus t, these values represent the slope of the tangent line to the graph in the point where, respectively $x(t) = 2$ and $x(t) = 4$. Note that these are the values of the variable x, which is represented on the *vertical* axis. This is demonstrated in Fig. 12.1. Since most people are used to considering a derivative as a function of the variable on the *horizontal* axis t, you may need some time to get used to this.

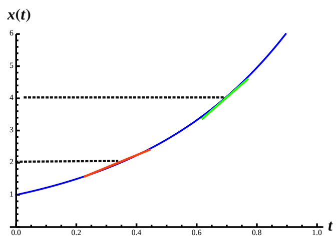


Figure 12.1: An example of the function $x(t)$ corresponding to the differential equation in (12.1). The red line is the tangent to the graph at $x = 2$, with slope 4, and the green line is the tangent at $x = 4$, with slope 8.

The general form of the type of differential equations that we are concerned with in this chapter is:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \tag{12.4}$$

In the example above, for instance, $f(x) = 2 \cdot x$. This is an example of a *linear model*, since $f(x)$ is a linear function of x. The general form of a *linear differential equation* is:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a \cdot x + b \tag{12.5}$$

An example of a non-linear model is the *logistic* model:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r \cdot x \left(1 - \frac{x}{K}\right), \text{ with } x > 0,\ r > 0,\ K > 0 \tag{12.6}$$

This model is used in population biology, where r is usually called the *intrinsic growth rate* and K the *carrying capacity*.

Exercises and assignments

1. Write the following differential equations in the form used in Eq. (12.1)

   a.
   $$x'(t) = 3x(t)^2 - 5x(t) + 1$$

   b.
   $$y'(t) = y(t) \cdot \left(1 - \frac{y(t)}{3}\right)$$

   c.
   $$x'(t) = a \cdot x(t) \cdot e^{-b \cdot x(t)}$$

2. Consider the following model:

   $$\frac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 - \frac{x}{10}\right)$$

   What is the instantaneous change in x at the following points?

   a. $x(t) = 10/2$

   b. $x(t) = 10$

   c. $x(t) = 20$

3. From the rules for derivatives it follows that if $z(t) = a \cdot x(t)$ then $z'(t) = a \cdot x'(t)$.

   a. Write this expression in terms of $\frac{\mathrm{d}z}{\mathrm{d}t}$ and $\frac{\mathrm{d}x}{\mathrm{d}t}$

   b. If it is given that $\frac{\mathrm{d}x}{\mathrm{d}t} = 2 \cdot x + 3$, what is the differential equation for $z = 2 \cdot x$?

12.2 *How do they work?*

To examine the mechanism of a differential equation, consider the definition of the differential quotient (derivative):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lim_{h \downarrow 0} \frac{x(t+h) - x(t)}{h} \tag{12.7}$$

Thus, for small values of h, the differential equation in (12.1) may be written as:

$$\frac{x(t+h)-x(t)}{h} \approx 2 \cdot x(t) \tag{12.8}$$

Rearranging this gives:

$$\begin{aligned} x(t+h)-x(t) &\approx 2 \cdot h \cdot x(t) \quad \text{multiply with } h \text{ on both sides} \\ x(t+h) &\approx x(t) + 2 \cdot h \cdot x(t) \quad \text{add } x(t) \text{ on both sides} \end{aligned} \tag{12.9}$$

Which is a linear recurrence relation, with time steps of length h and a multiplication factor of $(1 + 2 \cdot h)$. Thus, for small h, the differential equation specifies that the value of $x(t)$ grows with an amount of approximately $2 \cdot h$.

This expression provides a way to solve a differential numerically, through *Euler integration*. Choose a value for $x(0)$, and a small time step h, and iteratively calculate the values of $x(t)$, for example, if $x(0) = 1$ and $h = 0.1$, a value of 0.2 is added to $x(t)$ every time step, so the sequence of values is: $1, 1.2, 1.4, 1.6, \ldots$ If $x(0) = 2$, with the same value of h we get: $2, 2.2, 2.4, 2.6, \ldots$

These examples illustrate that the initial value $x(0)$ determines the outcome of a differential equation. The equation (12.1) by itself has an infinite number of possible solutions $x(t)$. Together with an *initial condition* $x(0)$, it specifies a unique outcome.

Euler integration may be used to solve differential equations numerically with Excel or other computer software. The approximation will get better as h decreases. Computer languages such as R contain programs with more sophisticated methods. The differential equation in (12.1) may, however, be solved explicitly by means of integration. In section 12.2.1 it is shown that the solution of a differential equation of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a \cdot x \tag{12.10}$$

equals:

$$x(t) = x(0)\, e^{at} \tag{12.11}$$

You may check this by taking derivatives on both sides and filling in the expression for x.

Thus, the solution of a linear differential equation such as (12.1) is an exponential function. This may come as a surprise to some people, because they tend to confuse this type of differential equation with an equation of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t \tag{12.12}$$

where the right-hand side is an expression in t. The solution of this differential equation is found by taking the antiderivative of the right-hand side, and filling in the initial condition:

$$x(t) = x(0) + t^2 \tag{12.13}$$

It is crucial to keep in mind the difference between these types of differential equations.

Although in some cases explicit solutions of differential equations can be found, in many practical situations this is not possible. We will turn to other ways of finding out what solutions $x(t)$ looks like in the following paragraphs. The solution of the linear equation provides a starting point for examining nonlinear differential equations, such as the logistic model in (12.6).

For instance, finding an exact solution of the so-called *Navier-Stokes* equations, that describe the motion of viscous fluid substances, is a remaining challenge in mathematics, and is listed as one of the Clay millennium problems (see urlhttp://www.claymath.org/millennium-problems).

Exercises and assignments

1. Consider the solution in (12.11)
   a. Compute the derivative of the expression for $x(t)$ in this equation.
   b. Show that the outcome is equal to $a \cdot x(t)$
2. Use R, Excel, or another device to compute the approximate solution of the differential equation $\frac{dx}{dt} = 2 \cdot x$, with $x(0) = 1$, by means of Euler integration. Compare the outcomes for $h = 0.1$ and $h = 0.01$ up to $t = 1$ with the real solution $x(t) = e^{2t}$.
3. Derive an approximation for the differential equation $\frac{dx}{dt} = 2 \cdot x + 1$ for small time steps h, using the same method as in (12.8).

12.2.1 *Solving a linear differential equation*

A differential equation of the form (12.10) may be solved explicitly by means of integration, as follows:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= a \cdot x \\
\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t} &= a && \text{divide by } x \text{ on both sides} \\
\int \frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t &= \int a\mathrm{d}t && \text{integrate both sides over } t \\
\int \frac{1}{x}\mathrm{d}x &= \int a\mathrm{d}t && \text{rewrite left} - \text{hand side} \\
\ln|x| &= at + c && \text{find antiderivatives}
\end{aligned}$$

If we fill in $x(0)$ we get:

$$\begin{aligned}
&\ln|x(0)| = c, \text{ so} \\
&\ln|x| = at + \ln|x(0)| \Rightarrow x(t) = x(0)\, e^{at}
\end{aligned} \tag{12.14}$$

12.3 *How do they behave?*

Solutions of univariate differential equations of the form (12.4) only show monotonic changes, i.e. $x(t)$ either continuously increases or decreases in time, without fluctuations.

The solution of the linear differential equation, $x(0) \cdot e^{at}$, increases and becomes infinitely large if $x(0) > 0$ and $a > 0$. If $x(0) < 0$ and $a > 0$ it decreases and becomes infinitely negative. If $a < 0$, e^{at}

In this respect, univariate differential equations differ remarkably from univariate recurrence relations, which may show cyclic or chaotic dynamics. See examples in section 10.3.

converges to zero for large values of t. In that case, $x(t)$ increases when the initial value lies below 0 and decreases when it lies above 0. These different types of solutions are shown in Fig. 12.2

The solutions of the logistic model in (12.6) converge to the carrying capacity K for all initial values $x(0) > 0$. This is shown in Fig. 12.3.

To summarise: solutions of univariate differential equations of the form in (12.4) always change monotonically in time. They either become infinitely large or negative, or they converge to a constant value. What happens depends on parameter values and initial conditions.

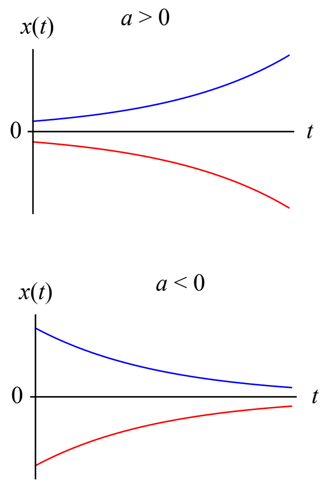


Figure 12.2: Shape of solutions of the linear differential equation in 12.10 for different combinations of a, and initial conditions $x(0) > 0$ (blue) and $x(0) < 0$ (red).

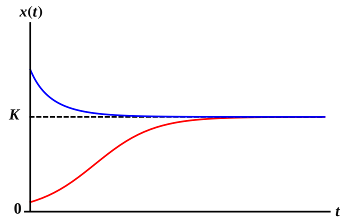


Figure 12.3: Shape of solutions of the logistic model in (12.6), for $x(0) < K$ (red) and $x(0) > K$ (blue).

12.4 *How can you tell?*

As shown in the previous section, dynamics of model solutions are affected by parameter values and/or initial conditions. In practical applications, the values of these quantities are unknown, or may vary, and usually we want to predict what happens for a wide range of circumstances. For relatively simple models, this might be done numerically, but, as models get more complicated, and contain more parameters, this is no longer feasible. In this section we will consider several different ways to predict model dynamics.

12.4.1 *Equilibria*

As in discrete models, equilibria are values of the state variable where no change occurs. This means that the derivative of x, $\frac{dx}{dt}$, is zero. For instance, in the models considered up to now (cf. Eqs. (12.6), (12.10)) the point 0 is an equilibrium. If a system starts in the equilibrium state, it will remain there forever. In other words, if $x(0)$ equals the equilibrium value, $x(t)$ will have the same value as $x(0)$ for all times.

This does not imply, however, that, if $x(0)$ does not equal the equilibrium value, $x(t)$ will move towards this. There are stable equilibria, which are attractors, and unstable ones, which are repellers. This is illustrated in Fig. 10.5. If the system is in an unstable equilibrium, the slightest perturbation will cause $x(t)$ to move away from that point, whereas stable equilibria are robust against such perturbations.

Equilibrium points play an important role in the mathematical analysis of dynamical systems. We will denote their values with a hat: $\hat{x}$, and add subscripts ($\hat{x}_1, \hat{x}_2$, etc.) when there are several such points. To find equilibria, equate $\frac{dx}{dt}$ to zero and solve the resulting *equilibrium equation*. For instance, for the general linear model:

$$\frac{dx}{dt} = a \cdot x + b \tag{12.15}$$

we get:

$$\begin{aligned} a \cdot \hat{x} + b &= 0 \\ \hat{x} &= -\frac{b}{a} \end{aligned} \tag{12.16}$$

The logistic model (Eq. 12.6) has two equilibrium points:

$$r \cdot \hat{x} \cdot \left(1 - \frac{\hat{x}}{K}\right) = 0 \Rightarrow \hat{x}_1 = 0, \hat{x}_2 = K \tag{12.17}$$

The general form of the equilibrium equation for continuous time models is:

$$f(\hat{x}) = 0, \tag{12.18}$$

where $f(x)$ defines the functional relationship between the derivative and the value of x (cf. (12.4)).

Exercises and assignments

1. What are the equilibrium points of the following model:

$$\frac{dx}{dt} = a \cdot x - \frac{b \cdot x}{1 + x}$$

2. The so-called Allee model for population dynamics is:

$$\frac{dx}{dt} = r \cdot x \cdot \left(\frac{x}{M} - 1\right) \cdot \left(1 - \frac{x}{K}\right), \text{with} x \geq 0, r > 0, M > 0, K > M \tag{12.19}$$

r and K are the intrinsic growth rate and the carrying capacity (just as in the logistic model), and M is called the minimum viable population size. What are the equilibria of this model?

3. The following model adds proportional harvesting at a rate h to the logistic model:

$$\frac{dx}{dt} = r \cdot x \cdot \left(1 - \frac{x}{K}\right) - h \cdot x, \text{ with } x \geq 0,\ r > 0,\ K > 0,\ h > 0 \tag{12.20}$$

   a. What is the equilibrium equation for this model?

   b. This model has at most two equilibria. What are their values?

   c. For which values of h does the model have a positive equilibrium?

12.4.2 *Types of equilibria*

The classification of equilibria can be further refined, by distinguishing different types of stability. Here, we will simply list the different types, and give a few examples. For a more extensive discussion, and clarifying figures, we refer to the section 10.4.2, where these concepts are explained in the context of discrete time models.

Dynamical models (whether they are continuous or discrete) may have the following types of equilibria:

- Unstable equilibria: these are repelling, no matter how close the initial value lies to the equilibrium point, the value of the state variable will move away from the equilibrium. Examples are the equilibrium $\hat{x} = 0$ in the linear model when $a > 0$, and the equilibrium $\hat{x} = 0$ in the logistic model (see Figs. 12.2 and 12.3).
- Globally stable equilibria: for (nearly) all initial values the system moves towards such a point. For instance the point $\hat{x} = K$ in the logistic model is globally stable, since for any value of $x(0) > 0$ the value of $x(t)$ moves towards this point. This is demonstrated in Fig. 12.3 for two different starting conditions. The dynamics are similar for other values of $x(0)$.
- Locally stable equilibria are attracting, but only for a limited range of initial conditions $x(0)$ this range of values is called the *region of attraction* of the equilibrium. This concept is illustrated in Fig. 10.6.
- Neutrally stable equilibria: these are attracting nor repelling. This concept is illustrated in Fig. 10.7. For example, if $a = 0$ in the linear model $\frac{dx}{dt} = a \cdot x$, all values of x are neutrally stable equilibrium points.
- Saddle points, or half stable equilibria: these are attracting from one side, and repelling from another side of the equilibrium (see Fig. 10.8).

Exercises and assignments

1. Consider the quadratic model:

$$\frac{dx}{dt} = 2 \cdot x^2 \tag{12.21}$$

   a. This model has one equilibrium. What is its value?
   b. What is the instantaneous change in x in this model at the points $x(t) = -1$ and $x(t) = 1$?
   c. When the instantaneous change in x is positive, the value of $x(t)$ increases in time, when it is negative, it decreases. What does this model predict concerning the value of $x(t)$ if you start at a value of $x(0) = -1$?
   d. What does this model predict concerning the value of $x(t)$ if you start at a value of $x(0) = +1$?
   e. For which values of $x(0)$ does $x(t)$ move towards the equilibrium and for which values does it move away?
   f. What type of equilibrium is $\hat{x}$?

2. Consider the quadratic model:

$$\frac{dx}{dt} = -2 \cdot x^2$$

   a. This model has one equilibrium. What is its value?

b. What is the instantaneous change in x in this model at the points $x = -1$ and $x = 1$?

c. What does this model predict concerning the value of $x(t)$ if you start at a value of $x(0) = -1$?

d. What does this model predict concerning the value of $x(t)$ if you start at a value of $x(0) = +1$?

e. What type of equilibrium is $\hat{x}$?

12.4.3 *Local stability*

As shown in the previous sections, the linear model $\frac{dx}{dt} = a \cdot x$ with $a \neq 0$ has the following properties:

- There is one equilibrium at $\hat{x} = 0$.
- $\hat{x}$ is (globally) stable if $a < 0$
- $\hat{x}$ is unstable if $a > 0$.
- The solution $x(t)$ changes monotonically in time.

The more general linear model $\frac{dx}{dt} = a \cdot x + b$ has an equilibrium at $\hat{x} = -\frac{b}{a}$. The properties of the equilibrium value depend on a in the same way as when $b = 0$. This can be shown by considering the transformation:

$$z(t) = x(t) + \frac{b}{a} \tag{12.22}$$

Since a and b are constants, we have the following result for the derivative:

$$z'(t) = x'(t) \Rightarrow \frac{dz}{dt} = \frac{dx}{dt} \tag{12.23}$$

so

$$\begin{aligned} \frac{dz}{dt} &= a \cdot x + b \\ &= a \cdot z + a\,(x - z) + b \end{aligned} \tag{12.24}$$

Since, by definition:

$$(x - z) = -\left(\frac{b}{a}\right) \tag{12.25}$$

we find:

$$\frac{dz}{dt} = a \cdot z \tag{12.26}$$

This is a linear model as before, with an equilibrium at 0, that is stable when $a < 0$ and unstable when $a > 0$.

For most nonlinear models it is not possible to find an explicit expression for the solution $x(t)$. Local properties of the dynamics near equilibrium points can, however, be studied by approximating the function $f(x)$ in (12.4) by a first order Taylor approximation. This amounts to the following approximation:

See section 19.4.

$$\frac{dx}{dt} \approx a \cdot x + b \tag{12.27}$$

where a and b are suitably chosen constants. From the previous results it follows that the local stability of the equilibrium point are determined by the value of a in the approximation. This value equals the derivative of $f(x)$ in the equilibrium point, $f'(\hat{x})$.

For example, consider the model:

$$\frac{dx}{dt} = x \cdot \left(1 - \frac{x}{3}\right) \tag{12.28}$$

There are two equilibrium points: $\hat{x}_1 = 0$ and $\hat{x}_2 = 3$. For this model we have:

$$f(x) = x \cdot \left(1 - \frac{x}{3}\right) \tag{12.29}$$

so the derivative is

$$f'(x) = 1 - \frac{2}{3}x \tag{12.30}$$

In the equilibrium point 0, the value of $f'(\hat{x})$ is 1. Since this is positive, the equilibrium is unstable. In the other equilibrium, we find $f'(\hat{x}_2) = 1 - 2 = -1$, so this is a stable equilibrium.

For the general logistic model:

$$f(x) = r \cdot x \cdot \left(1 - \frac{x}{K}\right) \tag{12.31}$$

and the derivative is (using the product rule):

$$\begin{aligned} f'(x) &= r \cdot \left(1 - \tfrac{x}{K}\right) - \tfrac{1}{K} \cdot r \cdot x \\ &= r \cdot \left(1 - 2\tfrac{x}{K}\right) \end{aligned} \tag{12.32}$$

Thus in the point $\hat{x}_1 = 0$:

$$f'(\hat{x}_1) = r \tag{12.33}$$

Since it is assumed that $r > 0$ in the logistic model, we can conclude that this equilibrium is unstable.

For the other equilibrium point $\hat{x}_2 = K$:

$$f'(\hat{x}_2) = -r \tag{12.34}$$

Since this is always negative in the logistic model, this equilibrium is stable.

In summary, for continuous time models we have the following results:

A more formal derivation is given in section 12.4.4.

- An equilibrium $\hat{x}$ is locally stable if $f'(x) < 0$.
- $\hat{x}$ is unstable if $f'(x) > 0$
- There are no oscillations: $x(t)$ changes monotonically in time.

These stability properties can be understood intuitively from the fact that the function $f(x)$ specifies the instantaneous change in x. At the equilibrium point $\hat{x}$, the change is zero, and therefore $f(\hat{x}) = 0$. If $f(x)$ increases in this point, this function must be negative on the left of $\hat{x}$ and positive on the right of his point. Therefore, on the left of the equilibrium point the change in x is negative, implying that x decreases, and moves away from $\hat{x}$. On the right-hand side of the equilibrium point, the change in x is positive, implying an increase, so again x moves away from the equilibrium point. Similarly, if $f(x)$ decreases in the point $\hat{x}$, it changes from positive to negative. In that case, x increases on the left of the equilibrium and decreases on its right-hand side, this implies x moves towards the equilibrium from both sides. This is illustrated if Fig. 12.4.

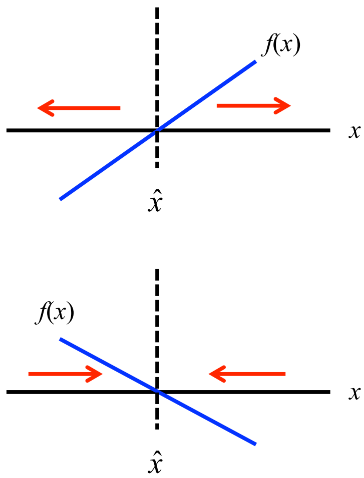


Figure 12.4: Illustration of local stability properties. Blue lines indicate the linear approximation of $f(x)$, with $f'(x) > 0$ in top figure, and $f'(x) < 0$ in bottom figure. Arrows indicate the directions of change in x.

If $f'(\hat{x}) = 0$ the first order approximation does not provide information concerning the local stability of the equilibrium. It turns out that in this case the properties depend on the value of second derivative $f''(\hat{x})$. If this value is positive, the equilibrium is a saddle point that is attracting from the left and repelling from the right hand side. If it is negative, the equilibrium is a saddle point that is attracting from the right and repelling from the left hand side. This is shown formally in section 12.4.4.

Exercises and assignments

1. Consider the following model:

$$\frac{dx}{dt} = x \cdot \left(\frac{x}{3} - 1\right) \cdot \left(1 - \frac{x}{20}\right) \tag{12.35}$$

   a. What is the function $f(x)$ in this case?
   b. What is the equilibrium equation?
   c. What are the equilibria of this model?
   d. What is the derivative $f'(x)$?
   e. Use the method presented in this section to determine which of the equilibria are locally stable, and which are unstable.

2. Consider the quadratic model:

$$\frac{dx}{dt} = a \cdot x^2 - h \cdot x \tag{12.36}$$

   a. What is the function $f(x)$ in this case?
   b. What is the equilibrium equation?
   c. What are the equilibria of this model?
   d. What is the derivative $f'(x)$?
   e. How does the stability of the equilibria depend on the values of a and h?
   f. When $h = 0$ there is a single equilibrium. What is its value?
   g. What type of equilibrium is this?

12.4.4 *Proof of local stability properties*

The results concerning local properties of equilibria can be formally derived as follows. The first order Taylor approximation of $f(x)$ in the point $\hat{x}$ is:

$$f(x) \approx f(\hat{x}) + f'(\hat{x})(x - \hat{x}) \tag{12.37}$$

Since $\hat{x}$ is an equilibrium point, $f(\hat{x}) = 0$, so:

$$f(x) \approx f'(\hat{x})(x - \hat{x}) \tag{12.38}$$

Therefore the approximation of the model

$$\frac{dx}{dt} = f(x)$$

is

$$\frac{dx}{dt} \approx f'(\hat{x})(x - \hat{x}) \tag{12.39}$$

Replacing x by $z = x - \hat{x}$ gives:

$$\frac{dz}{dt} \approx f'(\hat{x})\, z \tag{12.40}$$

This is a linear model of the form $\frac{dx}{dt}$, with parameter $a = f'(\hat{x})$, and the results follow from what is known about the solution of that model.

What happens if $f'(\hat{x}) = 0$?
In this case the first order approximation does not provide an answer concerning the local stability of $\hat{x}$, so we look at the second order approximation:

$$f(x) \approx f(\hat{x}) + f'(\hat{x})(x - \hat{x}) + \frac{1}{2} f''(\hat{x})(x - \hat{x})^2 \tag{12.41}$$

Since $f(\hat{x})$ and $f'(\hat{x})$ are both zero, we get:

$$f(x) \approx \frac{1}{2} f''(\hat{x})(x - \hat{x})^2 \tag{12.42}$$

and the approximating model becomes:

$$\frac{dx}{dt} \approx \frac{1}{2} f''(\hat{x})(x - \hat{x})^2 \tag{12.43}$$

Replacing x by $z = x - \hat{x}$ gives:

$$\frac{dz}{dt} \approx \frac{1}{2} f''(\hat{x})\, z^2 \tag{12.44}$$

This is a quadratic model, of the form:

$$\frac{dx}{dt} = b \cdot x^2 \tag{12.45}$$

This model has one equilibrium point at 0. When $b > 0$ the right-hand side of the equation is positive for all other values of x. This implies that $\frac{dx}{dt}$ is positive, so x increases for all initial values. This means that the equilibrium point 0 is a saddle point: if $x(0)$ is smaller than zero, it is attracting, and when $x(0)$ is larger than zero it is repelling. Similarly, when $b < 0$, the point 0 is a saddle point, since in this case x decreases in all other points. In this situation the equilibrium is attracting from the right-hand side and repelling from the left. This is illustrated if Fig. 12.5.

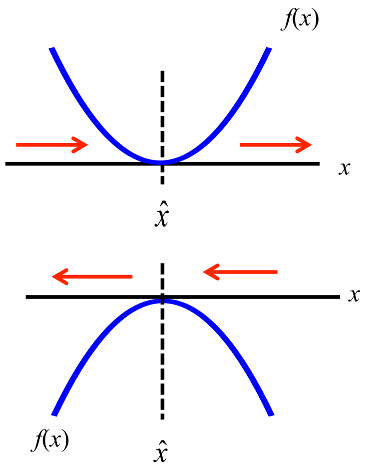


Figure 12.5: Illustration of stability properties of a quadratic model. Blue lines indicate $f(x) = b \cdot x^2$, with $b > 0$ in top figure, and $b < 0$ in bottom figure. Arrows indicate the directions of change in x.

12.4.5 Global analysis

Global properties, like regions of attraction of stable equilibria, can be studied by making a graph of the change $\frac{dx}{dt}$ versus x. For instance, the function that describes this relationship for the Allee model (see Eq. (12.19)) is a third order polynomial, with three $x - intercepts$: at $0, \ M$, and K. These points are the equilibrium points of the model. By studying the first derivative of $f(x)$, it can be shown that the function has the general shape shown in Fig. 12.6. Since it corresponds to the change in x, the state variable will decrease when $f(x) < 0$ and increase when $f(x) > 0$.
In the Allee model, choosing any value of $x(0)$ between 0 and M will cause $x(t)$ to move towards the stable equilibrium point 0. Any initial value above M will end up in $x(t)$ attaining the carrying capacity K.

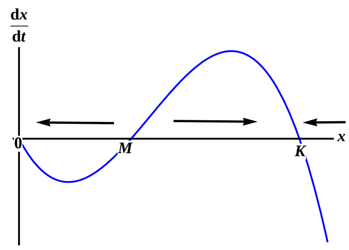


Figure 12.6: Graph of $\frac{dx}{dt}$ versus x for the Allee model. Arrows indicate directions of change in x.

This is the reason why M is called the *minimum viable population size* in ecological applications of this model.

Global analyses of continuous time models are simpler than those of discrete time models (see section 10.4.5), since in continuous time models, $x(t)$ is a continuous function of t. Whereas in univariate discrete time models, the state variable can jump from one side of an equilibrium point to the other side, this cannot happen with the type of models considered in this section.

Exercises and assignments

1. Make a graph of $\frac{dx}{dt}$ versus x for the logistic model (Eq. (12.6)). What is the region of atraction of the carrying capacity equilibrium?

2. Examine the effects of harvesting a logistic model, as in assignment 3. What happens to the graph of $\frac{dx}{dt}$ versus x as h increases? What does this imply for the population?

3. Make a graph of the following Allee model with harvesting:

$$\frac{dx}{dt} = x \cdot \left(\frac{x}{3} - 1\right) \cdot \left(1 - \frac{x}{10}\right) - h \cdot x, \text{ with } x \geq 0, h > 0 \quad (12.46)$$

Examine what happens when h increases, and discuss what this implies for the population.

13
Deterministic, continuous time, multivariate models

13.1 Linear, two-dimensional models

13.1.1 Notation

Multivariate models contain several state variables. We will use subscripts to denote those. Thus, in a two-dimensional model, there are two state variables, x_1 and x_2. The continuous time model is specified by two differential equations. Each of those describes the instantaneous change in one of the state variables. In a linear model, these equations are linear combinations of x_1 and x_2. For example:

$$\begin{aligned} \frac{dx_1}{dt} &= 2 \cdot x_1 - 3 \cdot x_2 \\ \frac{dx_2}{dt} &= 2 \cdot x_1 + 4 \cdot x_2 \end{aligned} \tag{13.1}$$

This model specifies that x_1 will increase when at time t the value $2 \cdot x_1$ is larger than $3 \cdot x_2$, and it will decrease when this inequality is reversed. The value of the other state variable, x_2 increases when at time t the inequality $2 \cdot x_1 > -4 \cdot x_2$ holds, and decreases when this is reversed.

The system of differential equations in (13.1) can be written in matrix notation as follows. Define:

An introduction to matrices and vectors is given in chapter 21.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \frac{d}{dt}\mathbf{x} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 2 & -3 \\ 2 & 4 \end{pmatrix} \tag{13.2}$$

then (13.1) corresponds to:

$$\frac{d}{dt}\mathbf{x} = \mathbf{A} \cdot \mathbf{x} \tag{13.3}$$

An often used alternative notation that means exactly the same is:

$$\dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x} \tag{13.4}$$

In general, a continuous time 2-dimensional linear model has the form:

$$\begin{aligned} \frac{dx_1}{dt} &= a \cdot x_1 + b \cdot x_2 \\ \frac{dx_2}{dt} &= c \cdot x_1 + d \cdot x_2 \end{aligned} \tag{13.5}$$

and the matrix **A** corresponds to:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{13.6}$$

Exercises and assignments

1. Write the following systems of equations in matrix notation:

   a.

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 + 3x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 2x_1 + x_2 \end{aligned} \tag{13.7}$$

   b.

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= -x_1 + x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1 - x_2 \end{aligned} \tag{13.8}$$

   c.

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= 2x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1 - x_2 \end{aligned} \tag{13.9}$$

13.1.2 *The equilibrium*

The equilibrium of a two-dimensional continuous time model is found by equating both differential equations to zero. This results in two equations for the state variables. For instance, for the model in (13.1) this gives:

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= 0 \Rightarrow 2 \cdot \hat{x}_1 - 3 \cdot \hat{x}_2 = 0 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 0 \Rightarrow 2 \cdot \hat{x}_1 + 4 \cdot \hat{x}_2 = 0 \end{aligned} \tag{13.10}$$

Solving these we find the equilibrium point $\hat{x}_1 = 0$ and $\hat{x}_2 = 0$.

Note that $\hat{x}_1$ and $\hat{x}_2$ are the coordinates of a single equilibrium point.

In general, for the linear model in (13.3), the equilibrium equation is:

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{0} \tag{13.11}$$

Exercises and assignments

1. Show that the model in (13.3) has only one equilibrium point, at $(0, 0)$.

2. Consider the following model:

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 + 3x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -2x_1 - 6x_2 \end{aligned}$$

a. Give the equilibrium equations.

b. How many equilibria are there?

c. What does the set of equilibria look like in a a graph with x_2 on the vertical axis, and x_1 on the horizontal axis?

3. Consider the general model in Eq. (13.5)

a. Write down the equilibrium equations for this model.

b. Show that when $a = 0$ and all other parameters are nonzero that $(0, 0)$ is the only equilibrium point.

c. What happens when $a = 0$ and $b = 0$?

d. Show that there are infinitely many equilibria when $a \cdot d - b \cdot c = 0$.

e. What does the set of equilibria look like in a a graph with x_2 on the vertical axis, and x_1 on the horizontal axis?

13.1.3 *The solution equation*

In section 12.2 it was shown that the solution of a one-dimensional linear differential equation is an exponential function of t. A similar result holds for multidimensional linear systems. We will state the main results for two dimensions here.

We do not include the derivations, since these are beyond the scope of this book, but the correctness of the results may be checked by differentiation (see exercises).

The form of the solutions of two-dimensional linear systems depend on the trace, T and determinant, D of the matrix $\mathbf{A}$. Provided that the determinant $D \neq 0$, the solutions can have two different forms: they are either combinations of exponential functions, or combinations of sine and cosine functions.

See section 21.3 for a definition of the trace and determinant.

If $D < \frac{1}{4}T^2$ the solutions are linear combinations of exponential functions:

$$\begin{aligned} x_1(t) &= \alpha \cdot e^{\lambda_1 t} + \beta \cdot e^{\lambda_2 t} \\ x_2(t) &= \gamma \cdot e^{\lambda_1 t} + \eta \cdot e^{\lambda_2 t} \end{aligned} \tag{13.12}$$

Here, α, β, γ and η are constants, whose values are determined by the initial condition, i.e. the values of $x_1(0)$ and $x_2(0)$ together with the so-called *eigenvectors* of the matrix $\mathbf{A}$. The quantities λ_1 and λ_2 are called the *eigenvalues* of $\mathbf{A}$. For a 2 by 2 matrix, their values can be calculated directly from its trace and determinant as follows:

Note the strong resemblance with the 'abc formula' for quadratic equations. See Eq. (18.11).

$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2} \tag{13.13}$$

With respect to the long term dynamics of the model solutions, the values of α, β, γ and η are irrelevant. Only the values of λ_1 and λ_2 determine whether or not the solutions converge to the equilibrium point $(0, 0)$. This happens if both of these values are negative. If at least one of them is positive, the distance between the state variables and the equilibrium point will increase exponentially in time.

Provided, of course, that they are not all equal to zero.

As an example consider the following model:

$$\frac{dx_1}{dt} = x_1 + 3 \cdot x_2$$
$$\frac{dx_2}{dt} = x_1 - x_2 \tag{13.14}$$

In this case, the matrix $\mathbf{A}$ is:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \tag{13.15}$$

With trace and determinant equal to $T = 0,\ D = -4$. From (13.13) we find $\lambda_1 = -2,\ \lambda_2 = 2$. Therefore, the solution equations have the following form:

$$x_1(t) = \alpha \cdot e^{-2t} + \beta \cdot e^{2t}$$
$$x_2(t) = \gamma \cdot e^{-2t} + \eta \cdot e^{2t} \tag{13.16}$$

It can be shown, for instance, that when the initial condition is $(1,1)$, the solution is:

$$x_1(t) = -\frac{1}{2}e^{-2t} + \frac{3}{2}e^{2t}$$
$$x_2(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{2t} \tag{13.17}$$

The first component of the solution, e^{-2t} will go to zero in the long run, so for large t we get:

$$x_1(t) \approx \frac{3}{2}e^{2t}$$
$$x_2(t) \approx \frac{1}{2}e^{2t} \tag{13.18}$$

Therefore, the values of both state variables will grow exponentially in the long run. This will happen for any choice of the initial condition, as long as it is not equal to the equilibrium point $(0,0)$.

If $D > \frac{1}{4}T^2$ the solutions are linear combinations of sine and cosine functions:

$$x_1(t) = e^{\lambda t}(\alpha \cdot \cos(\gamma \cdot t) - \beta \cdot \sin(\gamma \cdot t))$$
$$x_2(t) = e^{\lambda t}(\alpha \cdot \sin(\gamma \cdot t) - \beta \cdot \cos(\gamma \cdot t)) \tag{13.19}$$

The quantities α and β depend on the initial conditions and eigenvectors of $\mathbf{A}$. The quantities λ and γ correspond to:

$$\lambda = \frac{T}{2},\ \gamma = \frac{\sqrt{4D - T^2}}{2} \tag{13.20}$$

In this case, the values of the state variables will be cyclic, with a period length that depends on γ. With respect to long-term dynamics, however, only the value of λ is relevant: if it is negative, the amplitudes of the sine and cosine functions decrease in time, and the solution converges to the equilibrium point. If it is zero, the amplitudes remain constant, and the state variables will keep cycling around the equilibrium value. If it is positive, the fluctuations will increase with time, and the average distance to the equilibrium point will increase.

As an example, consider the following model:

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -x_1 - x_2\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 4x_1 - x_2\end{aligned} \tag{13.21}$$

The matrix **A** for this model is:

$$\mathbf{A} = \begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix} \tag{13.22}$$

so: $T = -2,\ D = 5$. From (13.20) we find: $\lambda = -1,\ \gamma = 2$, and thus the solution has the following form:

$$\begin{aligned}x_1(t) &= e^{-t}\left(\alpha \cdot \cos(2t) - \beta \cdot \sin(2t)\right)\\ x_2(t) &= e^{-t}\left(\alpha \cdot \sin(2t) - \beta \cdot \cos(2t)\right)\end{aligned} \tag{13.23}$$

The solution for the initial condition $(1, 1)$ for instance, can be shown to be:

$$\begin{aligned}x_1(t) &= e^{-t}\left(\cos(2t) - \frac{1}{2} \cdot \sin(2t)\right)\\ x_2(t) &= e^{-t}\left(2 \cdot \sin(2t) + \cos(2t)\right)\end{aligned} \tag{13.24}$$

As t increases, the multiplication factor e^{-t} converges to zero, and thus both state variables become zero in the long run.

To conclude, in a two-dimensional system, the properties of the equilibrium and the dynamics of the state variables can be directly inferred from the trace T and the determinant D. In the following section we will examine the different possible situations.

Exercises and assignments

1. Use Excel, R, or a graphing calculator to draw graphs of the functions $x_1(t)$ and $x_2(t)$ in (13.17) and their approximations in (13.18).

2. Use Excel, R, or a graphing calculator to draw graphs of the functions $x_1(t)$ and $x_2(t)$ in (13.24).

3. Consider the solution in (13.17).

   a. Calculate $\frac{\mathrm{d}x_1}{\mathrm{d}t}$ by differentiating the right hand side of the first equation.

   b. Show that the outcome is equal to $x_1(t) + 3 \cdot x_2(t)$.

   c. Calculate $\frac{\mathrm{d}x_2}{\mathrm{d}t}$ by differentiating the right hand side of the second equation.

   d. Show that the outcome is equal to $x_1(t) - x_2(t)$.

   e. What can you conclude from these results?

4. Consider the solution in (13.24).

a. Calculate $\frac{dx_1}{dt}$ by differentiating the right hand side of the first equation.

b. Show that the outcome is equal to $-x_1(t) - x_2(t)$.

c. Calculate $\frac{dx_2}{dt}$ by differentiating the right hand side of the second equation.

d. Show that the outcome is equal to $4x_1(t) - x_2(t)$.

e. What can you conclude from these results?

13.1.4 Dynamics

As stated previously, the linear model in (13.3) has a single equilibrium point at $(0, 0)$, unless the determinant of the matrix $\mathbf{A}$ equals zero. The type and stability of this equilibrium point are determined by the properties of $\mathbf{A}$, more specifically by its trace T and determinant D. The main characteristics are:

- The equilibrium is stable of $D > 0$ and $T < 0$
- The solutions fluctuate if $D > \frac{1}{4}T^2$
- The solutions are monotonic if $D \leq \frac{1}{4}T^2$

Figure 13.1 gives a summary of all the different types of dynamics for the different combinations of values of D and T.

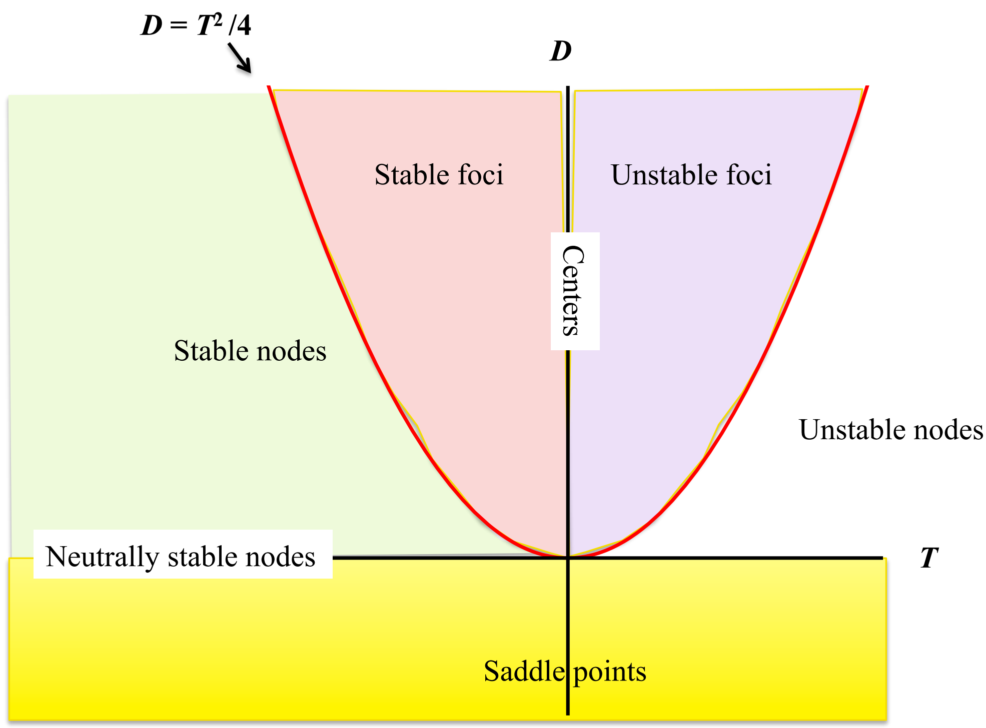


Figure 13.1: Overview of the different types of dynamics of the linear two-dimensional model, for different combinations of D and T. See text below for further explanation.

We will give an overview of the different types of equilibria and the corresponding dynamics below, with some examples. To illustrate the dynamics, we will show two types of graphs. One graph is the usual one, with the values of the state variables $x_1(t)$ and $x_2(t)$ versus time, t. The other graph shows $x_1(t)$ on the horizontal axis, and the corresponding value of $x_2(t)$ on the vertical axis. This type of plot is called a *phase plot*. Phase plots provide additional insight in the joined dynamics of the state variables, for instance to detect whether there are stable cycles. The direction in which variables change are indicated by arrows in the phase plots.

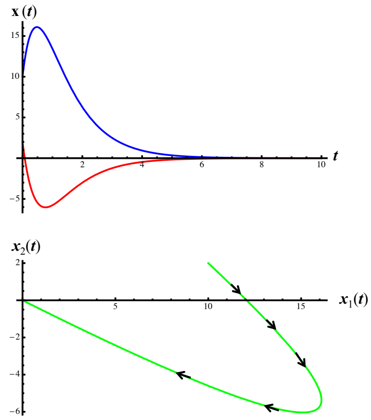


Figure 13.2: Dynamics of the linear model with **A** given in (13.25) and initial condition $(10, 2)$. Top: $x_1(t)$(blue) and $x_2(t)$(red) versus time. Bottom: phase plot.

Stable equilibria: $D > 0$, and $T < 0$

- ***Stable node***: $T < 0, 0 < D \leq \frac{1}{4}T^2$ Regardless of the initial condition, the solutions converge to the equilibrium. There are no fluctuations
  Example:

$$\mathbf{A} = \begin{pmatrix} 2 & 6 \\ -2 & -5 \end{pmatrix} \tag{13.25}$$

  Figure 13.2 shows an example. The top figure shows the dynamics of both state variables in time.

  As demonstrated in the figure, the solutions may initially move away from the equilibrium point. This depends on the initial conditions. After such an initial increase, however, the distance to the equilibrium decreases monotonically.

- ***Stable focus***: $T < 0, D > \frac{1}{4}T^2$ In this case, the state variables fluctuate, and fluctuations become smaller with time.
  Example

$$\mathbf{A} = \begin{pmatrix} -0.1 & 1 \\ -1 & -0.1 \end{pmatrix} \tag{13.26}$$

  Figure 13.3 shows an example of the dynamics of this model. Note that, since this equilibrium is a focus, there are fluctuations, but they are not so obvious. For other initial conditions they may be more evident.

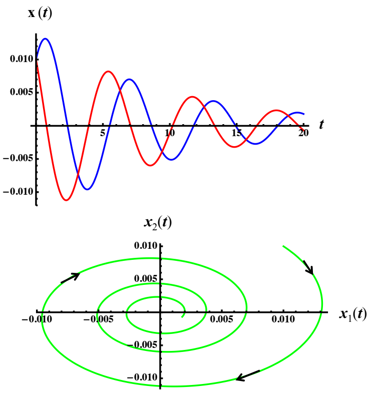


Figure 13.3: Dynamics of the linear model with **A** given in (13.26) and initial condition $(0.01, 0.01)$. Top: $x_1(t)$(blue) and $x_2(t)$(red) versus time. Bottom: phase plot.

Neutrally stable equilibria $T = 0$ and $D > 0$, or $D = 0$

- ***Center***: $T = 0,\ D > \frac{1}{4}T^2$
  The solutions fluctuate with constant amplitude. They move neither toward, nor away from the equilibrium. Their amplitude and average distance to the equilibrium are determined by the initial condition.
  Example

$$\mathbf{A} = \begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix} \tag{13.27}$$

  An example of the dynamics is shown in Fig. 13.4. Note that another initial condition will lead to a different cycle (unless it

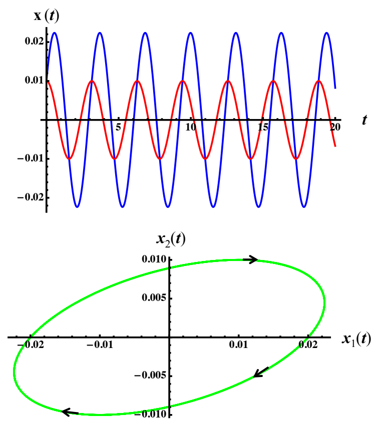


Figure 13.4: Dynamics of the linear model with **A** given in (13.27) and initial condition $(0.01, 0.01)$. Top: $x_1(t)$(blue) and $x_2(t)$(red) versus time. Bottom: phase plot.

happens to lie exactly on the same trajectory as in the figure). Also, should a disturbance occur, the system will assume a different cyclic trajectory.

- ***Neutrally stable nodes***: $D = 0$
  If $D = 0$ the model has infinitely many equilibria. The long term outcome then depends on the initial condition. We will not consider such situations further here.

Unstable equilibria: $T > 0$, and/or $D < 0$

- ***Saddle point***: $D < 0$
  The equilibrium is a *saddle point* . The equilibrium is unstable, but it is attracting along the direction of one of the eigenvectors, and repelling along the direction of the other eigenvector.
  Example:

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \tag{13.28}$$

Figures 13.5 and 13.6 show examples of the dynamics of this model, for two different initial conditions: in 13.5 the initial condition is chosen in such a way that the solution converges to the equilibrium, in the other figure it is such that the solution moves away from the equilibrium. Note that, even though the solutions may, in special cases, tend towards the equilibrium, this is an unstable equilibrium point, since (almost) any disturbance will cause the state variables to move away from the equilibrium.

- ***Unstable node***: $T > 0, 0 < D \leq \frac{1}{4}T^2$
  Regardless of the initial condition, the solutions go away from the equilibrium, in a monotonic fashion.
  Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \tag{13.29}$$

Figure 13.7 shows an example of the dynamics for this model. As demonstrated, the divergence from the equilibrium happens very quickly. Since the solutions are exponential functions, this may be expected.

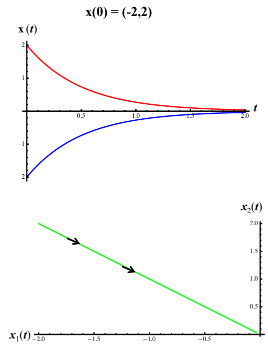


Figure 13.5: Dynamics of the linear model with $\mathbf{A}$ given in (13.28) and an initial condition that leads to the equilibrium. Top: $x_1(t)$(blue) and $x_2(t)$(red) versus time. Bottom: phase plot.

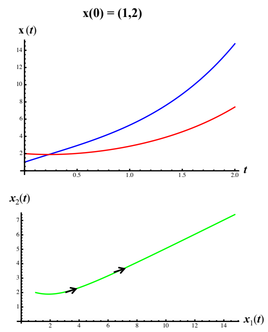


Figure 13.6: Dynamics of the linear model with $\mathbf{A}$ given in (13.28) and an initial condition that leads away from the equilibrium. Top: $x_1(t)$(blue) and $x_2(t)$(red) versus time. Bottom: phase plot.

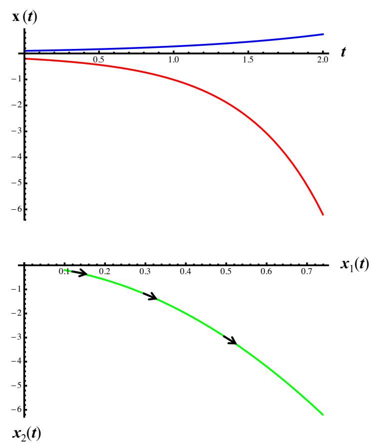


Figure 13.7: Dynamics of the linear model with $\mathbf{A}$ given in (13.29) and initial condition $(0.1, -0.2)$. Top: $x_1(t)$(blue) and $x_2(t)$(red) versus time. Bottom: corresponding phase plot.

- ***Unstable focus***: $T > 0, D > \frac{1}{4}T^2$
  The solutions cycle, and move away to the equilibrium. Fluctuations become larger with time.
  Example

$$\mathbf{A} = \begin{pmatrix} 0.1 & 1 \\ -1 & 0.1 \end{pmatrix} \tag{13.30}$$

Figure 13.8 shows an example of the dynamics.

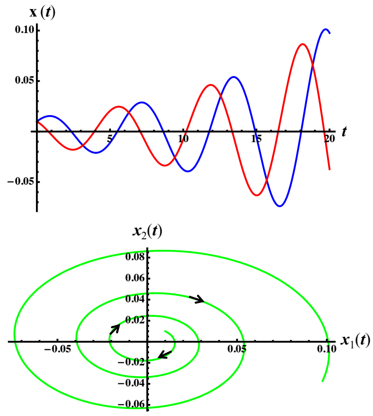


Figure 13.8: Dynamics of the linear model with $\mathbf{A}$ given in (13.30) and initial condition $(0.01, 0.01)$. Top: $x_1(t)$(blue) and $x_2(t)$(red) versus time. Bottom: phase plot.

Exercises and assignments

1. Determine the type of the equilibrium in each of the following models, based on the trace and determinant of the matrix $\mathbf{A}$:

   a.
$$\frac{dx_1}{dt} = 2x_1 + 3x_2$$
$$\frac{dx_2}{dt} = x_1 + 2x_2$$

   b.
$$\frac{dx_1}{dt} = 3x_2$$
$$\frac{dx_2}{dt} = x_1$$

   c.
$$\frac{dx_1}{dt} = -x_1 + -2x_2$$
$$\frac{dx_2}{dt} = x_1 - x_2$$

2. Determine how the stability of the equilibrium in each of the following models depends on p:

   a.
$$\frac{dx_1}{dt} = 2p \cdot x_1 + 3x_2$$
$$\frac{dx_2}{dt} = x_1 + 2x_2$$

   b.
$$\frac{dx_1}{dt} = x_1 - px_2$$
$$\frac{dx_2}{dt} = x_1$$

3. Determine for which values of p the equilibrium of the following model is a stable focus:

$$\frac{dx_1}{dt} = -x_1 - px_2$$
$$\frac{dx_2}{dt} = x_1$$

13.1.5 *The general linear model*

The more general linear model has the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{A} \cdot \mathbf{x} + \mathbf{b} \tag{13.31}$$

where $\mathbf{b}$ is a column vector of constants. An example is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} \cdot \mathbf{x} + \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{13.32}$$

This model may also be written as:

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 - 2x_2 + 1 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 3x_1 + 2x_2 + 4 \end{aligned} \tag{13.33}$$

We will show that this type of models can be transformed to the previously considered ones, and thus very similar conclusions can be drawn concerning their dynamics. The equilibrium equation for the model is:

$$\mathbf{A} \cdot \hat{\mathbf{x}} + \mathbf{b} = \mathbf{0} \tag{13.34}$$

This corresponds to two equations with two unknowns, $\hat{x}_1$ and $\hat{x}_2$. For instance, for the model in (13.32) we get:

$$\begin{aligned} \hat{x}_1 - 2\hat{x}_2 + 1 &= 0 \\ 3\hat{x}_1 + 2\hat{x}_2 + 4 &= 0 \end{aligned} \tag{13.35}$$

with solution $\hat{x}_1 = -0.8, \hat{x}_2 = 0.1$.

We can write (13.31) in a different way, as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{A} \cdot (\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{b} + \mathbf{A} \cdot \hat{\mathbf{x}} \tag{13.36}$$

and, using the equilibrium equation (13.34) it follows that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{A} \cdot (\mathbf{x} - \hat{\mathbf{x}}) \tag{13.37}$$

Define:

$$\mathbf{z} = (\mathbf{x} - \hat{\mathbf{x}}) \tag{13.38}$$

then, since $\hat{\mathbf{x}}$ is a vector of constants, its derivative is $\mathbf{0}$, and:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x} - \hat{\mathbf{x}}) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} \tag{13.39}$$

Therefore, (13.37) can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z} = \mathbf{A} \cdot \mathbf{z} \tag{13.40}$$

which corresponds to the linear model that we have studied before, with as state variable the distance to the equilibrium. Therefore, all the conclusions concerning the relationship between the trace and determinant of $\mathbf{A}$ and the type and stability of the equilibrium continue to hold for the more general model in (13.31). The only aspect that changes is the location of the equilibrium point.

Exercises and assignments

1. Consider the following model:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 2x_1 + 3x_2 - 1$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 - 4x_2 + 3$$

   a. What are the matrix **A**, and the vector **b** in this case?
   b. What are the equations for the equilibrium?
   c. What are the coordinates of the equilibrium?
   d. What are the trace and determinant of **A**?
   e. What type of equilibrium is this?

2. Answer the same questions as in the previous exercise, for the model:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 4x_1 - x_2 + 1$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_1 + x_2$$

13.2 *Nonlinear, two-dimensional models*

In a nonlinear model one or both differential equations for the changes in state variables are nonlinear. An example of such a model is:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 3x_1 - 2x_1 \cdot x_2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 \cdot x_2 - x_2 \tag{13.41}$$

A general notation for such a model is:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1, x_2)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1, x_2) \tag{13.42}$$

In the example of (13.41) these functions correspond to:

$$f_1(x_1, x_2) = 3x_1 - 2x_1 \cdot x_2$$
$$f_2(x_1, x_2) = x_1 \cdot x_2 - x_2 \tag{13.43}$$

Multivariate models may also be represented in matrix notation, as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{F}(\mathbf{x}) \tag{13.44}$$

or

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{13.45}$$

where **x** denotes a vector of the state variables, and **F** is a vector-valued function:

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} \tag{13.46}$$

This notation clearly shows the analogy with univariate models.

Just as with one-dimensional models, nonlinear multivariate models may be approximated by linear models, and the dynamics in the vicinity of equilibria can be studied accordingly.

Exercises and assignments

1. Identify the functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ for each of the following models:

   a.
$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 2{x_1}^2 + 3x_1 \cdot (x_2 - 1)\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1\end{aligned}$$

   b.
$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \frac{x_1}{1+2x_1} - x_2\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 2x_1 + x_2\end{aligned}$$

13.2.1 Equilibria

As before, equilibria are points in which no change occurs. In the multivariate case, that means that all state variables should maintain the same value. Thus, $\frac{\mathrm{d}x_i}{\mathrm{d}t}$ must be zero for all i, and the equilibrium equation is:

$$\mathbf{F}(\hat{\mathbf{x}}) = \mathbf{0} \tag{13.47}$$

which is completely analogous to the univariate case. Thus, the equilibria are found by solving the system of equations:

$$\begin{aligned}f_1(\hat{x}_1, \hat{x}_2) &= 0\\ f_2(\hat{x}_1, \hat{x}_2) &= 0\end{aligned} \tag{13.48}$$

For the example in (13.41), this gives:

$$\begin{aligned}3\hat{x}_1 - 2\hat{x}_1 \cdot \hat{x}_2 &= 0\\ \hat{x}_1 \cdot \hat{x}_2 - \hat{x}_2 &= 0\end{aligned} \tag{13.49}$$

The first equation gives:

$$3\hat{x}_1 - 2\hat{x}_1 \cdot \hat{x}_2 = 0 \Rightarrow \hat{x}_1 = 0, \text{or } 3 - 2\hat{x}_2 = 0 \Rightarrow \hat{x}_2 = \frac{3}{2} \tag{13.50}$$

Subsituting $\hat{x}_1 = 0$ in the second equation gives $\hat{x}_2 = 0$, so $(0, 0)$ is an equilibrium point. Substituting $\hat{x}_2 = \frac{3}{2}$ in the second equation gives:

$$\hat{x}_1 \cdot \frac{3}{2} - \frac{3}{2} = 0 \Rightarrow \hat{x}_1 = 1 \tag{13.51}$$

so the other equilibrium point is $\left(1, \frac{3}{2}\right)$

Exercises and assignments

1. What are the equilibrium points of the following models?:

   a.

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 - 3x_1x_2 \\ \frac{dx_2}{dt} &= 2x_1x_2 - x_2\end{aligned} \tag{13.52}$$

   b.

$$\begin{aligned}\frac{dx_1}{dt} &= 4x_1\left(1 - \frac{x_1}{2}\right) - 3x_1x_2 \\ \frac{dx_2}{dt} &= 3x_1x_2 - 2x_2\end{aligned} \tag{13.53}$$

13.2.2 *Local properties*

The multivariate analogue of the linear approximation in eqs. (12.39) and (12.40) is:

$$\begin{aligned}\frac{d}{dt}\mathbf{x} &\approx \mathbf{F}'(\hat{\mathbf{x}}) \cdot (\mathbf{x} - \hat{\mathbf{x}}) \\ \frac{d}{dt}\mathbf{z} &\approx \mathbf{F}'(\hat{\mathbf{x}}) \cdot \mathbf{z}\end{aligned} \tag{13.54}$$

where $\mathbf{z}$ denotes the vector of distances to the equilibrium point, $\mathbf{x} - \hat{\mathbf{x}}$. The matrix $\mathbf{F}'(\hat{\mathbf{x}})$ is the *Jacobian matrix* (also called *Jacobian*) of $\mathbf{F}(\mathbf{x})$, evaluated at the point $\hat{\mathbf{x}}$. This is the matrix of partial derivatives of the functions in $\mathbf{F}$ with respect to the variables x_i.

For the example in (13.41), for instance, the partial derivatives are:

$$\begin{aligned}\frac{\partial f_1}{\partial x_1} &= 3 - 2x_2, & \frac{\partial f_1}{\partial x_2} &= -2x_1 \\ \frac{\partial f_2}{\partial x_1} &= x_2, & \frac{\partial f_2}{\partial x_2} &= x_1 - 1\end{aligned} \tag{13.55}$$

Thus, in the equilibrium point $(0, 0)$:

$$\mathbf{F}'(\hat{\mathbf{x}}) = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \tag{13.56}$$

The trace of this matrix is $T = 2$ and the determinant -3, so this equilibrium is a saddle point.

See subsection 13.1.4.

Exercises and assignments

1. What are the trace and the determinant of the Jacobian matrix for the other equilibrium point, $(1, 3/2)$ in the example? What type of equilibrium is this?

2. Consider the model in (13.52)

   a. What is the Jacobian matrix for this model?

b. What is the Jacobian, evaluated at the equilibrium $(0,0)$?
c. What type of equilibrium is this?
d. What type of equilbrium is the point $(1/2, 1/3)$?

3. Consider the model in (13.53).

a. What is the Jacobian matrix for this model?
b. What type of equilibrium is $(0,0)$?
c. What type of equilibrium is $(2,0)$?
d. What type of equilibrium is $(2/3, 8/9)$?

13.3 *Higher-dimensional models*

The analysis of models with three or more state variables is similar to that described for two-dimensional models in the previous sections. Solving the equilibrium equation (13.47) may be much harder, especially if the model's dimension is large. In many cases one must resort to numerical (computer) methods for finding the equilibria. The same holds for analysis of local dynamics near equilibria. As in two-dimensional models, the stability of an equilibrium point is determined by the properties of the Jacobian matrix, evaluated at that point. As opposed to the 2-dimensional case, though, there is no simple connection with quantities that can easily be computed, such as the trace and the determinant. Only in rare cases it is possible to examine this algebraically when there are more than three state variables. Therefore, numerical methods, available in computer packages, are used.

The type of dynamics that continuous time models may exhibit depends on their dimensions. Univariate models can only show monotonic dynamics: the state variable either increases, or decreases in time. Therefore, the only possible long-term behaviour for univariate models is that the state variable converges to an equilibrium point, or becomes infinitely positive or negative. As shown in the current chapter, in two-dimensional continuous time models, fluctuations may occur. Here too, convergence to equilibria is possible, but, in addition there may be sustained limit cycles. In models with three or more state variables, other dynamics are also possible. These cannot simply be described as cycles, or other geometric objects, and are called *strange attractors*.

Note that in this respect there is a fundamental difference between continuous time and discrete time models. As shown in chapter 11, in discrete time cyclic or chaotic dynamics can already occur in univariate models.

For example, the so-called Lorentz system of differential equations, which is a model for the earth's atmosphere:

$$\begin{aligned}\frac{dx_1}{dt} &= a \cdot (x_2 - x_1) \\ \frac{dx_2}{dt} &= c \cdot x_1 - x_1 \cdot x_3 - x_2 \\ \frac{dx_3}{dt} &= x_1 \cdot x_2 - b \cdot x_3 \end{aligned} \tag{13.57}$$

shows such dynamics for specific parameter combinations. This is illustrated in Fig. 13.9.

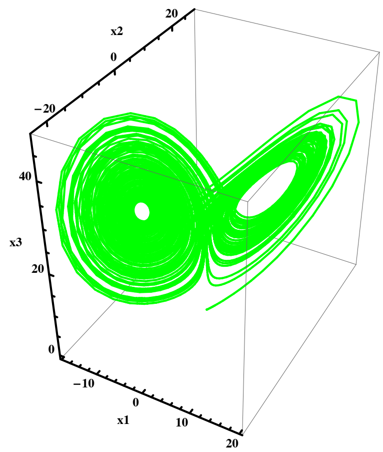


Figure 13.9: Strange attractor of the Lorentz system in (13.57) with $a = 10, b = 8/3, c = 28$.

14
Answers to exercises of Part II

Chapter 9 Preliminaries

section 9.2

1. a How does the graph of y in the example of Eq. (9.4) look when $a = 0$?
   answer:
   Then the equation states: $y = b$, and the plot corresponds to a horizontal line.

   b What is the solution x of the equation $a \cdot x + b = 0$ when $a = 0$?
   answer:
   Then the equation states $b = 0$. If the parameter b is not zero, there is no solution x, if $b = 0$ all values of x are solutions.

section 9.3.1

1. The unit of force is the Newton. Expressed in SI units this is kg m s^{-2}. What is the dimension of force, expressed in the basic physical quantities of Table 9.1?
   answer:
   mass $\cdot$ length $\cdot$ time^{-2}
2. As was stated above, Watt is Joule per second, and, as a consequence, its dimension is energy per time. In a similar way the relation between force and energy may be examined
   a. What is the relation between Newtons and Joules?
      answer:
      A Newton is Joule per metre.
   b. Based on this, what is the dimensional relation between force and energy?
      answer:
      Force is energy per distance (length). Equivalently: energy is force times length.
3. Acceleration is the increase of velocity per time unit.
   a. What is the (SI) unit of acceleration?
      answer:
      meter per second squared
   b. What is the relation between acceleration and force?
      answer:
      Force is mass times acceleration. Equivalently: acceleration is force divided by mass.
4. The *magnetic flux density* is a measure for the strength of a magnetic field. Its unit is the Tesla, which corresponds to Newton$/(\mathrm{A} \cdot \mathrm{m})$.
   a. What is the dimension of this quantity?
      answer:
      It is force/(electrical current $\cdot$ length), which equals mass/(time^2 $\cdot$ current)

b. What is its relation with energy?
answer:
Since the dimension of energy is mass · length^2/time^2, the magnetic flux density has dimension energy/(current · length^2).

section 9.3.2

1. The density of water is 1 gram/cm^3.How much does 2.51 m^3 of water weigh in kilograms?
answer: 1 m^3 = 10^6 cm^3, so it weighs $2.5 \cdot 10^6$ grams, which is $2.5 \cdot 10^3$ kg.
2. Every year, about 20 million litres of water are discharged into a river, by a storm drain. The runoff is contaminated: storm drain water contains about 150 micrograms of a toxic heavy metal per litre. How many grams of this metal are dumped into the river per year?
answer: The total discharge is $20 \cdot 10^6 \text{litre} \cdot 150 \frac{\text{microgram}}{\text{litre}} = 3 \cdot 10^9 \text{microgram} = 3 \cdot 10^6 \text{gram}$.
3. A tanker caused a fuel oil spill with a layer of 4.5 inches, over an area of 190 acres. An inch is 2.54 cm and acre is 4.0469 m^2. How many litres of oil were spilled?
answer: The total volume spilled is:

$$\begin{aligned} &4.5 \text{ inches} \cdot 2.54 \frac{cm}{inch} 190 \text{ acres} \cdot 4.0469 \frac{\text{m}^2}{\text{acres}} = 8788.65 \text{ cm} \cdot \text{m}^2 \\ &= 8788.65 \text{ cm} \cdot \text{m}^2 \cdot 10^4 \frac{\text{cm}^2}{\text{m}^2} = 8788.65 \cdot 10^4 \text{cm}^3 \\ &= 8788.65 \cdot 10^4 \text{cm}^3 \cdot \frac{1}{10^3} \frac{\text{litre}}{\text{cm}^3} \approx 87887 \text{ litre} \end{aligned}$$

metre

section 9.3.3

1. Show that (9.13) gives the same velocity after 30 seconds as in (9.17).
answer:
Eq. (9.13) gives
$$v(30) = 9.8 \cdot 30 = 294 \frac{\text{m}}{\text{s}}$$
which equals $294 \cdot \frac{60}{1000} = 17.64 \frac{\text{km}}{\text{min}}$.
2. Express the outcome of (9.19) in km/hr.
answer:
$166.9 \frac{\text{km}}{\text{min}} \cdot 60 \frac{\text{min}}{\text{hr}} = 10014 \frac{\text{km}}{\text{hr}}$
3. What is the dimension of the parameter a in (9.21)?
answer:
Since x has dimension 1/length^3, a must have this dimension too.
4. A model for the thickness h of a slab of ice in the ocean, in relation to temperature states:
$$h(t) = \sqrt{{h_0}^2 + \frac{2k(T_w - T_a)t}{L \cdot D}}$$
where $h(t)$ is its thickness at time t, h_0 the initial thickness, T_w the water temperature, and T_a the air temperature. k is the thermal conductivity of the ice, D the density of sea water (its weight per volume), and L its *latent heat*. This is the amount of energy that is released per mass when sea water turns into ice.

   a. What is the dimension of $h(t)$?
   answer:

b. What must be the dimension of the second expression under the square root sign, for the model to be consistent?
answer:
The same as ${h_0}^2$, so length^2

c. What is the dimension of $L \cdot D$?
answer:
L has dimension energy/mass and D has dimension mass/volume, so the product has dimension energy/volume. Expressed in the basic dimensions, energy has dimension mass length^2 time^{-2}, so the dimension of $L \cdot D$ is: mass time^{-2} length^{-1}.

d. What must be the dimension of the heat conductivity, k, to make the model dimensionally consistent?
answer:
The total expression $\frac{2k(T_w - T_a)t}{L \cdot D}$ must have dimension length^2, so If we denote the dimension of k by $[k]$ the second expression under the square root sign has dimension:

$$\frac{[k] \cdot \text{temperature} \cdot \text{time}}{\left(\frac{\text{mass}}{\text{time}^2 \text{length}}\right)} = [k] \cdot \frac{\text{temperature} \cdot \text{time}^3 \cdot \text{length}}{\text{mass}}$$

This must be equal to length^2, so

$$[k] \cdot \frac{\text{temperature} \cdot \text{time}^3 \cdot \text{length}}{\text{mass}} = \text{length}^2$$
$$[k] = \frac{\text{mass} \cdot \text{length}^2}{\text{temperature} \cdot \text{time}^3 \cdot \text{length}} = \frac{\text{mass} \cdot \text{length}}{\text{temperature} \cdot \text{time}^3}$$

e. k is usually expressed in the units $\frac{\text{Watt}}{\text{K} \cdot \text{m}}$. Show that the corresponding dimension agrees with your answer.
answer:
The dimension of Watt is energy/time, which corresponds to mass length^2 time^{-2}. Divided by temperaturelength this gives a total dimension of mass length time^{-3}temperature^{-1} which is indeed the same as found before.

Chapter 10
Deterministic, discrete time, univariate models

section 10.1

1. Consider the general model formulation of Eq. (10.3).
   a. What is the function $f(x)$ for the Beverton-Holt model?
   answer: $f(x) = \frac{ax}{1+bx}$
   b. This is one of the basic functions described in chapter 17. Which?
   answer: It is a linear fractional function.
   c. Is the function in the Beverton-Holt model increasing or decreasing in x?
   answer: Increasing: this can be inferred from the description of the linear fractional functions in chapter 17, or determined by means of the derivative.
   d. What is the horizontal asymptote of this function?
   answer: $y = a/b$
   e. Sketch a graph of this function.
   answer: This is a monotonically increasing function, starting at $(0, 0)$. For large x it converges to $y = a/b$.

2. Consider the general model formulation of Eq. (10.3).
   a. What is the function $f(x)$ for the Ricker model?
   answer: $f(x) = axe^{-bx}$
   b. What is the value of this function at $x = 0$?
   answer: $f(0) = 0$
   c. What happens to the value of $f(x)$ as x becomes (infinitely) large, and what does this imply for the graph of the function?
   answer: It goes to zero. There is a horizontal asymptote at $y = 0$.
   d. This function has one extremum, at which value of x does it occur?
   answer: $f'(x) = ae^{-bx} - abxe^{-bx}$, this is zero when $x = 1/b$.
   e. Is this a minimum or a maximum?
   answer: It is a maximum. You can see it by determining the second derivative: $f''(x) = -2abe^{-bx} + ab^2xe^{-bx}$, $f''(1/b) = -2ab + ab = -ab < 0$. Alternatively, it follows from the fact that $f(0) = 0$, the asymptote is at zero, and $f(x)$ is positive when $x > 0$.
   f. What does a graph of this function look like?
   answer: This is a function that starts at $(0, 0)$, with a maximum at $x = 1/b$. It goes to zero as x increases further.

3. Write down the general form of a quadratic model. How many parameters are there?
   answer: $x(n+1) = ax^2 + bx + c$. The model has three parameters.

4. Consider the Beverton-Holt model, with $a = 2$ and $b = 1$.
   a. What is the value of $x(n+1)$ if $x(n) = 0.5$?
      answer: $2/3$
   b. What is the value of $x(n+1)$ if $x(n) = 2$?
      answer: $4/3$
   c. Can you think of a value of $x(n)$ that would give the exactly same value for $x(n+1)$?
      answer: two values would give this result: 0 or 1 (one suffices)
5. Consider the Ricker model, with $a = 2$ and $b = 1$.
   a. What is the value of $x(n+1)$ if $x(n) = 1$?
      answer: $2/e$
   b. What is the value of $x(n+1)$ if $x(n) = 2$?
      answer: $4/e^2$
   c. Can you think of a value of $x(n)$ that would give exactly the same value for $x(n+1)$?
      answer: two values would give this result: 0 or $\ln(2)$ (one suffices)
6. Write the Beverton-Holt model in the form of a difference equation, as was done in (10.8) for the quadratic model.
   answer: $x(n+1) - x(n) = \frac{a \cdot x(n)}{1+b \cdot x(n)} - x(n)$
7. Reformulate the linear model in (10.4) as a recurrence relation between $x(n)$ and $x(n-1)$
   answer: $x(n) = a \cdot x(n-1) + b$

section 10.2

1. Consider the model $x(n+1) = 0.2 \cdot x(n) + 1$.
   a. This is a model of the form given in (10.16). What is the value of $b/(1-a)$ in this case?
      answer: $1/0.8 = 10/8$
   b. The solution equation for the general model is given in Eq. (10.23). What is it in this specific case?
      answer: $x(n) = (10/8) + (0.2^n) \cdot (x(0) - 10/8)$.
   c. What is the value of $x(n+1)$ if $x(n) = b/(1-a)$?
      answer: $x(n) = 10/8 (= b/(1-a))$
   d. What is the value of $x(n)$ if $x(0) = b/(1-a)$?
      answer: $x(n) = 10/8 (= b/(1-a))$
2. Consider the model $x(n+1) = 0.5 \cdot x(n) - 2$.
   a. This is a model of the form given in (10.16). What is the value of $b/(1-a)$ in this case?
      answer: $-2/0.5 = -4$
   b. What is its solution equation?
      answer: $x(n) = -4 + (0.5^n) \cdot (x(0) + 4)$
   c. What is the value of $x(n+1)$ if $x(n) = b/(1-a)$?
      answer: $x(n) = -4 (= b/(1-a))$
   d. What is the value of $x(n)$ if $x(0) = b/(1-a)$?
      answer: $x(n) = -4 (= b/(1-a))$
3. Consider the general expression for the linear model, in Eq. (10.16).
   a. What is the value of $x(n+1)$ if $x(n) = b/(1-a)$?
      answer: $x(n+1) = b/(1-a)$
   b. What is the value of $x(n)$ if $x(0) = b/(1-a)$?
      answer: $x(n) = b/(1-a)$

4. Consider the model $x(n+1) = \gamma \cdot x(n) - \beta$.
   a. This is a model of the form given in (10.16). What is the value of $b/(1-a)$ in this case?
   answer: $-\frac{\beta}{\gamma-1}$
   b. What is its solution equation?
   answer: $x(n) = -\frac{\beta}{\gamma-1} + \gamma^n \cdot (x(0) + \frac{\beta}{\gamma-1})$
5. Calculate the values of $x(1)$ and $x(2)$ for the Beverton-Holt model with $a = 0.2,\ b = 1,\ x(0) = 1$.
   answer: $x(1) = 0.1, x(2) \approx 0.018$
6. Calculate the values of $x(1)$ and $x(2)$ for the Ricker model with $a = 0.2,\ b = 0.1,\ x(0) = 1$.
   answer: $x(1) \approx 0.18, x(2) \approx 0.036$
7. Derive the solution equation of the following models, using the same method as in (10.17).
   a. $x(n+1) = 3 \cdot x(n) + 2$
   answer: In this model $b/(1-a) = 2/(-2) = -1$. Subtracting -1 on both sides of the model equation gives. $x(n+1) + 1 = 3 \cdot x(n) + 3$, so $(x(n+1) + 1) = 3 \cdot (x(n) + 1)$. Thus, if we define $z(n) = x(n) + 1, z(n+1) = 3 \cdot z(n) \Rightarrow z(n) = 3^n \cdot z(0)$. It follows that $x(n) + 1 = 3^n \cdot (x(0) + 1)$ and therefore $x(n) = -1 + 3^n \cdot (x(0) + 1)$.
   b. $x(n+1) = 2 \cdot x(n) - 5$
   answer: In this model $b/(1-a) = -5/(-1) = 5$. Subtracting 5 on both sides of the model equation gives. $x(n+1) - 5 = 2 \cdot x(n) - 10$, so $(x(n+1) - 5) = 2 \cdot (x(n) - 5)$. Thus, if we define $z(n) = x(n) + 1, z(n+1) = 2 \cdot z(n) \Rightarrow z(n) = 2^n \cdot z(0)$. It follows that $x(n) + 1 = 2^n \cdot (x(0) - 5)$ and therefore $x(n) = 5 + 2^n \cdot (x(0) - 5)$.
8. Derive the solution of the linear model for the special case that $a = 1$, by writing down the expressions for $x(1), x(2), ...$ and finding a pattern, as was done in (10.12). What is the solution equation? What kind of relationship does this model predict between $x(n)$ and n?
   answer: $x(1) = x(0) + b$, $x(2) = x(1) + b = x(0) + 2b, \ldots, x(n) = x(0) + n \cdot b$. A linear relationship.

section 10.3

Use Excel, R, or a graphics calculator for the following assignments.

1. Replicate the different types of dynamics of the Ricker model (see Eq. (10.6)) that were shown in this section. Choose your own value of b.
2. Examine the dynamics of the linear model for different values of a, and a non-zero value of b.
3. Examine the dynamics of the linear model when $a = 1$ and a value of $b \neq 0$.
4. Examine the dynamics of the Beverton-Holt model (see (10.5)). Compare what happens when $a \leq 1$ and when $a > 1$. Also explore the effect of varying b.

section 10.4.1

1. Determine the equilibria of the quadratic model in (10.25).
   answer: $\hat{x}_1 = 0$ and $\hat{x}_2 = 2$.
2. How many equilibria can a general quadratic model have at most?
   answer: Two, since the equilibrium equation is a quadratic equation, it can have at most two solutions.
3. Adding proportional harvesting at a rate of h to a Beverton-Holt model gives the following recurrence equation:

$$x\left(n+1\right) = \frac{2 \cdot x\left(n\right)}{1 + x\left(n\right)} - h \cdot x\left(n\right),\ x \geq 0,\ a > 0,\ b > 0,\ h \geq 0 \tag{14.1}$$

a. This model may have at most two equilibria. What are their values?
answer: $\hat{x}_1 = 0$, $\hat{x}_2 = \frac{1-h)}{(1+h)}$

b. How large may h maximally be for the second equilibrium to exist?
answer: $h < 1$

4. Adding the same type of harvesting to a Ricker model gives:

$$x\,(n+1) = 1.5 \cdot x(n) \cdot e^{-0.1x(n)} - h \cdot x(n) \tag{14.2}$$

a. This model may have at most two equilibria. What are their values?
answer: $\hat{x}_1 = 0, \hat{x}_2 = 10 \cdot \ln\left(\frac{1.5}{1+h}\right)$

b. For which range of h-values does the model have two equilibria?
answer: When $1.5 > 1 + h$, so $h < 0.5$.

5. Consider the Beverton-Holt model with harvesting, with general parameter values:

$$x\,(n+1) = \frac{a \cdot x\,(n)}{1 + b \cdot x\,(n)} - h \cdot x\,(n)\,,\; x \geq 0,\; a > 0,\; b > 0,\; h \geq 0 \tag{14.3}$$

a. What are the expressions for the two equilibria?
answer: $\hat{x}_1 = 0$, $\hat{x}_2 = \frac{a-1-h)}{b \cdot (1+h)}$

b. Assume that $a > 1$, so there are two equilibria if $h = 0$. How large may h maximally be to keep two equilibria when there is harvesting?
answer: $h < a - 1$

6. Answer the same questions for the Ricker model with harvesting and general parameter values.
answer: The model becomes $x\,(n+1) = a \cdot e^{-b \cdot x(n)} - h \cdot x(n)$.
equilibria : $\hat{x}_2 = \frac{1}{b} \cdot \ln\left(\frac{a}{1+h}\right)$
the model has two equilibria if $a > 1 + h$, so $h < a - 1$

section 10.4.2

1. Use Excel, R, or another device to calculate a few values of $x(n)$ for the quadratic model in (10.34). Choose starting values close to 0, on either side of the equilibrium.

2. Use Excel, R, or another device to examine the stability of the equilibrium at 0 for the model:

$$x(n+1) = x(n) \cdot (x(n) - 1) \tag{14.4}$$

section 10.4.3

1. Consider the following model:

$$x\,(n+1) = \frac{2 \cdot x\,(n)}{1 + 5 \cdot x\,(n)} \tag{14.5}$$

a. What is the function $f(x)$ that describes the relationship between $x(n+1)$ and $x(n)$ for this model?
answer:

$$f\,(x) = \frac{2 \cdot x}{1 + 5 \cdot x}$$

b. This model has two equilibria, what are their values?
answer: $\hat{x}_1 = 0, \hat{x}_2 = 1/5$

c. What is the derivative $f'(x)$?
answer:

$$f'(x) = \frac{2}{(1 + 5 \cdot x)^2}$$

d. What are the values of $f'(\hat{x}_1)$ and $f'(\hat{x}_2)$?
answer: $f'(\hat{x}_1) = 2$, $f'(\hat{x}_2) = 1/2$

e. Which of the equilibria is stable, and which is unstable?
answer: $\hat{x}_1$ is unstable, $\hat{x}_2$ is stable.

f. Are the local dynamics near equilibria monotonic or not?
answer: Yes

2. Consider the general Beverton-Holt model of Eq. (10.5).

a. What is the function $f(x)$ that describes the relationship between $x(n+1)$ and $x(n)$ for this model?
answer: $f(x) = \frac{a \cdot x}{1+b \cdot x}$

b. What is the derivative $f'(x)$?
answer:

$$f'(x) = \frac{a \cdot (1 + b \cdot x) - b \cdot a \cdot x}{(1 + b \cdot x)^2}$$
$$= \frac{a}{(1 + b \cdot x)^2}$$

c. For which parameter combinations is the equilibrium $\hat{x}_1 = 0$ stable?
answer: $f'(0) = a$, so it is stable if $0 < a < 1$ (remember: $a > 0$ in this model).

d. Are the dynamics near the equilibrium monotonic or fluctuating?
answer: Monotonic, since $a > 0$.

e. Examine the stability of the other equilibrium point, $\hat{x}_2 = \frac{a-1}{b}$.
answer: $f'(\hat{x}_2) = \frac{1}{a}$ So the second equilibrium of this model is stable when $a > 1$.

f. What can you say about the long term dynamics of this model if there are two equilibria?
answer: It was already noted before that the second equilibrium only exists when $a > 1$, since the state variable must be positive in this model. So the second equilibrium is stable when it exists, and the first equilibrium is unstable in this case.

3. The equilibria of the Ricker model of Eq, (10.6) were previously shown to be $\hat{x}_1 = 0$ and $\hat{x}_2 = \frac{\ln(a)}{b}$.

a. What is the function $f(x)$ that describes the relationship between $x(n+1)$ and $x(n)$ for this model?
answer: $f(x) = a \cdot x \cdot e^{-bx}$

b. What is the derivative $f'(x)$?
answer: $f'(x) = a \cdot e^{-bx} - b \cdot a \cdot x \cdot e^{-bx}$

c. Use a first order approximation to determine for which parameter values the first equilibrium is stable, and for which it is unstable.
answer: $f'(0) = a$, the equilibrium is stable if $0 < a < 1$. The model specifies that a is positive. It is unstable of $a > 1$.

d. Use a first order approximation to determine for which parameter values the second equilibrium is stable or unstable.
answer: $f'(\frac{\ln(a)}{b} = 1 - \ln(a)$. This equilibrium is stable if $-1 < 1 - \ln(a) < 1$, so $0 < \ln(a) < 2$, $1 < a < e^2$

e. What can you say about the type of dynamics around the second equilibrium: is it monotonic or fluctuating?
answer: If $1 - \ln(a) < 0$ there are fluctuations, so when $a > e$.

f. What is the value of the second derivative of $f(x)$ in the point $\hat{x}_1 = 0$, and what does this mean for the stability of this equilibrium? (Note: the state variable must be positive in this model!)
answer: $f''(0) = -2ab = -2b$ if $a = 1$.This equilibrium is a saddle point, which is attracting from the right-hand side. Since $x(0)$ cannot be negative, this means the equilibrium is stable.

g. Compare the results of your local analysis with the dynamics for this model, that were shown in Fig. 10.3.
answer: The results in this section 10.3 show that when $a < 1$, $x(n)$ goes to zero. When $e < a$ there are indeed fluctuations. The state variable $x(n)$ converges to the second equilibrium, so the fluctuations decrease. When $e^2 < a$, there is no convergence to the equilibrium anymore. Instead, there are stable limit cycles, or chaos.

section 10.4.5

Use the cobweb method in combination with local analysis and numerical methods to find out the dynamics of the following models for several combinations of parameter values (choose your own values):

1. $x(n+1) = \frac{a \cdot x(n)}{1+b \cdot x(n)} - h \cdot x$, with $x \geq 0$ and $a, b, h > 0$
2. $x(n+1) = a \cdot x(n) \cdot e^{-b \cdot x(n)} - h \cdot x$, with $x \geq 0$ and $a, b, h > 0$

Chapter 11
Deterministic, discrete time, multivariate models

section 11.1.1

1. What are the values of the two state variables at time $n+1$ in the model of Eq. (11.3), if $x_1(n) = 2$, and $x_2(n) = 1$?
   answer: $x_1(n+1) = 1,\ x_2(n+1) = 8$
2. What is the matrix $\mathbf{A}$ in each of the following cases?

   a.
$$x_1(n+1) = 5 \cdot x_1(n) + x_2(n)$$
$$x_2(n+1) = 2 \cdot x_1(n) - 2 \cdot x_2(n)$$

   answer:
$$\mathbf{A} = \begin{pmatrix} 5 & 1 \\ 2 & -2 \end{pmatrix}$$

   b.
$$x_1(n+1) = x_1(n) + x_2(n)$$
$$x_2(n+1) = x_1(n) - x_2(n)$$

   answer:
$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

   c.
$$x_1(n+1) = x_2(n)$$
$$x_2(n+1) = x_1(n) - x_2(n)$$

   answer:
$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

3. For each of the models in the previous question, give the values of the state variables at times 1 and 2, if the initial condition is $x_1(0) = 1, x_2(0) = 1$.
   answer:

   a.
$$x_1(1) = 6, x_2(1) = 0$$
$$x_1(2) = 30, x_2(2) = 12$$

b.
$$x_1(1) = 2, x_2(1) = 0$$
$$x_1(2) = 2, x_2(2) = 2$$

c.
$$x_1(1) = 1, x_2(1) = 0$$
$$x_1(2) = 0, x_2(2) = 1$$

section 11.1.2

1. Write down the equilibrium equations for the following model, and demonstrate that $(0,0)$ is the only outcome:

$$x_1(n+1) = 5 \cdot x_1(n) + x_2(n)$$
$$x_2(n+1) = 2 \cdot x_1(n) - 2 \cdot x_2(n)$$

   answer:
   The equilibrium equations are

$$\hat{x}_1 = 5 \cdot \hat{x}_1 + \hat{x}_2$$
$$\hat{x}_2 = 2 \cdot \hat{x}_1 - 2 \cdot \hat{x}_2$$

   The first equation implies $-4 \cdot \hat{x}_1 = \hat{x}_2$, so $\cdot \hat{x}_1 = -1/4 \cdot \hat{x}_2$. The second equation gives $2 \cdot \hat{x}_1 = 3 \cdot \hat{x}_2$, so both are satisfied if $-1/4 \cdot \hat{x}_2 = 3 \cdot \hat{x}_2$ which means $\hat{x}_2$ must be zero. This in turn implies that $\hat{x}_1 = 0$

2. Consider the more general model representation in (11.5).
   a. What are the equilibrium equations?
   answer:

$$\hat{x}_1 = a \cdot \hat{x}_1 + b \cdot \hat{x}_2$$
$$\hat{x}_2 = c \cdot \hat{x}_1 + d \cdot \hat{x}_2$$

   b. Show that, when $a = 1$ and all other parameters are non-zero the only equilibrium point is $(0,0)$.
   answer: When $a = 1$, the first equation states $\hat{x}_1 = \hat{x}_1 + b \cdot \hat{x}_2$, so if $b \neq 0$ $\hat{x}_2$ must be zero. Substituting this value in the second equation gives $\hat{x}_1 = 0$.
   c. What are the equilibria when $a = 1$, $c = 0$, and b and d are non-zero?
   answer: Then it follows from the second equation that $\hat{x}_2 = 0$ and substitution in the first equation gives $\hat{x}_1 = \hat{x}_1$, since this is always true, $\hat{x}_1$ may have any value. There are infinitely many equilibria in this case: any point with coordinate $x_2 = 0$ is an equilibrium.

section 11.1.3

1. For each of the following models, give the trace and determinant of the matrix **A**:
   a.
$$x_1(n+1) = \frac{1}{2} \cdot x_1(n) - x_2(n)$$
$$x_2(n+1) = x_1(n) + 3 \cdot x_2(n)$$

   answer: trace:7/2, determinant:5/2
   b.
$$x_1(n+1) = -x_2(n)$$
$$x_2(n+1) = 2x_1(n) + 5 \cdot x_2(n)$$

   answer: trace:5, determinant:2

c.
$$x_1(n+1) = -2 \cdot x_1(n) + x_2(n)$$
$$x_2(n+1) = x_1(n) - x_2(n)$$

answer: trace:-3, determinant:1

d.
$$x_1(n+1) = x_2(n)$$
$$x_2(n+1) = x_1(n)$$

answer: trace:0, determinant:-1

section 11.1.4

1. Examine the dynamics of the solutions for the different examples numerically, using different initial conditions than those in the figures.
2. Determine the type of equilibrium for the linear models corresponding to the following matrices **A**:

a.
$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$

answer: $T = 1$, so $T - 1 = 0, -T - 1 = -2, \frac{1}{4}T^2 = 1/4$, and $D = -2$
$D = -T - 1$, and $D < T - 1$, $D < \frac{1}{4}T^2$ The equilibrium is an unstable node.

b.
$$\begin{pmatrix} -1 & 1 \\ 1/2 & 0 \end{pmatrix}$$

answer: $T = -1$, so $T - 1 = -2, -T - 1 = 0, \frac{1}{4}T^2 = 1/4$, and $D = -1/2$
$D < -T - 1$, and $D > T - 1$, $D < \frac{1}{4}T^2$ The equilibrium is a saddle point.

c.
$$\begin{pmatrix} 1/2 & 1 \\ -3 & 0 \end{pmatrix}$$

answer: $T = 1/2$, so $T - 1 = -1/2, -T - 1 = -3/2, \frac{1}{4}T^2 = 1/16$, and $D = 3$
$D > T - 1, D > -T - 1, D > \frac{1}{4}T^2$. The equilibrium is an unstable spiral

d.
$$\begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}$$

answer: $T = 2$, so $T - 1 = 1, -T - 1 = -3, \frac{1}{4}T^2 = 2$, and $D = 3$
$D > 1$ and $D > \frac{1}{4}T^2$. The equilibrium is an unstable spiral.

3. Determine for which values of the parameter α the equilibrium of the linear models corresponding to the following matrices **A** is stable:

a.
$$\begin{pmatrix} \alpha & 2 \\ 1 & 0 \end{pmatrix}$$

answer:
$D = -2$, so the equilibrium is unstable for all values of α.

b.
$$\begin{pmatrix} 1 & \alpha \\ 1 & 0 \end{pmatrix}$$

answer:
$T = 1,\ D = -\alpha$. The equilibrium is stable if $0 < D < 1$ so if $-1 < \alpha < 0$.

c.

$$\begin{pmatrix} 0 & -\alpha \\ 1 & 1 \end{pmatrix}$$

answer:
$T = 1,\ D = \alpha$. The equilibrium is stable if $0 < D < 1$, so if $0 < \alpha < 1$.

4. Construct a matrix $\mathbf{A}$ with $D = T - 1$ or $D = -T - 1$, and examine the dynamics of the corresponding linear model.
   answer:
   For instance a matrix with $T = 0$ and $D = -1$, such as for instance

$$\mathbf{A} = \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$$

   Note, however, that there are many other correct answers.

section 11.1.5

1. Compute the trace and determinant of the matrix $\mathbf{A}$ for the model in (11.20). What can you say concerning the model dynamics? Check your predictions numerically, using several different initial conditions.
   answer:
   $T = 3, D = 3$ so the equilibrium is an unstable spiral. Solutions spiral away from the equilibrium. Fluctuations of the solutions increase in time.

section 11.2

1. What are the values of the state variables at times 1 and 2 for the model in (11.26), with initial condition $x_1(0) = 1$ and $x_2(0) = 1$?
   answer:

$$x_1(1) = \frac{3}{2},\ x_2(1) = -\frac{2}{3}$$
$$x_1(2) = 2 \cdot \frac{3}{2} - \frac{1}{2} \cdot \frac{3}{2} \cdot \left(-\frac{2}{3}\right) = 3 + \frac{1}{2} = \frac{7}{2},$$
$$x_2(1) = \frac{1}{3} \cdot \frac{3}{2} + \frac{2}{3} = \frac{7}{6}$$

2. What are the functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ in the following model?

$$x_1(n+1) = 2x_1(n) \cdot (1 - x_2(n))$$
$$x_2(n+1) = x_1(n)$$

   answer:

$$f_1(x_1, x_2) = 2x_1 \cdot x_2$$
$$f_2(x_1, x_2) = x_1$$

3. What are the functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ in the following model?

$$x_1(n+1) = x_1(n) \cdot \left(1 - \frac{x_1(n)}{K}\right) - \alpha \cdot x_1(n) \cdot x_2(n)$$
$$x_2(n+1) = \lambda \cdot \alpha \cdot x_1(n) \cdot x_2(n) - \mu \cdot x_2(n)$$

   answer:

$$f_1(x_1, x_2) = x_1 \cdot \left(1 - \frac{x_1}{K}\right) - \alpha \cdot x_1 \cdot x_2$$
$$f_2(x_1, x_2) == \lambda \cdot \alpha \cdot x_1 \cdot x_2 - \mu \cdot x_2$$

section 11.2.1

1. Show that $(0,2)$ and $(6,0)$ are not equilibrium points of the model in (11.26).
   answer:
   If $x_1 = 0$ and $x_2 = 2$, Eq. (11.28) gives $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = -2$, which is not equal to $(0,2)$. Similarly, with $x_1 = 6$ and $x_2 = 0$, we get $f_1(x_1, x_2) = 12$ and $f_2(x_1, x_2) = 0$, which is not equal to $(6,0)$. Therefore, neither of these points satisfy the equilibrium equations in (11.32).
2. Consider the following model:

$$x_1(n+1) = x_1(n) \cdot x_2(n) - 2 \cdot x_1(n)$$
$$x_2(n+1) = x_2(n) \cdot (2 - x_2(n)) - 3 \cdot x_1(n) \cdot x_2(n)$$

   a. Give the equilibrium equations for this model.
   answer:

$$\hat{x}_1 = \hat{x}_1 \cdot \hat{x}_2 - 2 \cdot \hat{x}_1$$
$$\hat{x}_2 = \hat{x}_2 \cdot (2 - \hat{x}_2) - 3 \cdot \hat{x}_1 \cdot \hat{x}_2$$

   b. Give the coordinates of the three equilibrium points for this model.
   answer: The first equilibrium equation gives:

$$\hat{x}_1 = \hat{x}_1 \cdot \hat{x}_2 - 2 \cdot \hat{x}_1 \Rightarrow \hat{x}_1 = 0, \text{ or } 1 = \hat{x}_2 - 2 \Rightarrow \hat{x}_2 = 3$$

   Filling in $\hat{x}_1 = 0$ in the second equation gives

$$\hat{x}_2 = \hat{x}_2 \cdot (2 - \hat{x}_2) \Rightarrow \hat{x}_2 = 0, \text{ or } 1 = (2 - \hat{x}_2) \Rightarrow \hat{x}_2 = 1$$

   So this results in two equilibria: $(0,0)$ and $(0,1)$. Filling in $\hat{x}_2 = 3$ in the second equation gives:

$$3 = 3 \cdot (2-3) - 3 \cdot \hat{x}_1 \cdot 3 \Rightarrow 3 = -3 - 9 \cdot \hat{x}_1 \Rightarrow \hat{x}_1 = -\frac{6}{9} = -\frac{2}{3}$$

   so the third equilibrium point is $(-\frac{2}{3}, 3)$.

11.2.2

1. What is the Jacobian matrix for the model in (11.26), evaluated at the other equilibrium point, $(3,4)$, and what type of equilibrium is this?
   answer: The Jacobian in this equilibrium is:

$$\mathbf{J}(\hat{\mathbf{x}}) = \begin{pmatrix} 0 & -\frac{3}{2} \\ \frac{4}{3} & 0 \end{pmatrix}$$

   So $T = 0, D = 2$. The equilibrium is an unstable spiral.
2. Consider the following model:

$$x_1(n+1) = -x_1(n) + \frac{1}{2} x_1(n) \cdot x_2(n)$$
$$x_2(n+1) = -\frac{1}{2} x_1(n) \cdot x_2(n) + 5 x_2(n)$$

   a. What are the functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ for this model?
   answer:

$$f_1(x_1, x_2) = -x_1 + \frac{1}{2} x_1 \cdot x_2$$
$$f_2(x_1, x_2) = -\frac{1}{2} x_1 \cdot x_2 + 5 x_2$$

b. Show that $(0,0)$ is an equilibrium point of this model.
answer: Filling in 0 for each of the state variables on the right-hand side gives 0 as an outcome on the left hand side for both equations.

c. Find the coordinates of the second equilibrium point for this model.
answer: $(8,4)$

d. Determine the Jacobian matrix for this model.
answer:

$$\mathbf{J}(\mathbf{x}) = \begin{pmatrix} -1+\frac{1}{2}x_2 & \frac{1}{2}x_1 \\ -\frac{1}{2}x_2 & -\frac{1}{2}x_1+5 \end{pmatrix}$$

e. Use a first order approximation to predict the local dynamics of the state variable near the equilibrium (0,0). What type of equilibrium is this?
answer:

$$\mathbf{J}(\hat{\mathbf{x}}) = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

$T = 4, D = -5$. The equilibrium is an unstable node.

f. Do the same for the other equilibrium point.
answer:

$$\mathbf{J}(\hat{\mathbf{x}}) = \begin{pmatrix} 1 & 4 \\ -2 & 1 \end{pmatrix}$$

$T = 2, D = 7, \frac{1}{4}T^2 = 1$. It is an unstable spiral.

3. Consider the following model:

$$x_1(n+1) = \frac{1}{2}x_1(n)\cdot x_2(n) - 2\cdot x_1(n)$$
$$x_2(n+1) = x_2(n)\cdot\left(1-\frac{x_2(n)}{3}\right) - x_1(n)\cdot x_2(n)$$

a. Show that $(0,0)$ is an equilibrium point of this model.
answer: Filling in 0 for each of the state variables on the right-hand side gives 0 as an outcome on the left hand side for both equations.

b. Determine the Jacobian matrix for this model.
answer:

$$\begin{pmatrix} \frac{1}{2}x_2 - 2 & \frac{1}{2}x_1 \\ -x_2 & 1-\frac{2}{3}x_2 - x_1 \end{pmatrix}$$

c. Use a first order approximation to predict the local dynamics of the state variable near the equilibrium $(0,0)$. What type of equilibrium is this?
answer:
Evaluating the Jacobian at the equilibrium $(0,0)$ gives:

$$\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

$T = -1, D = -2$, so $T-1 = -2, -T-1 = 0$. Thus, $D = T-1, D < -T-1$ The equilibrium is an unstable node.

Chapter 12
Deterministic, continuous time, univariate models

section12.1

1. Write the following differential equations in the form used in Eq. (12.1)

   a.
   $$x'(t) = 3x(t)^2 - 5x(t) + 1$$

   answer:
   $$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x^2 - 5x + 1$$

   b.
   $$y'(t) = y(t) \cdot \left(1 - \frac{y(t)}{3}\right)$$

   answer:
   $$\frac{\mathrm{d}y}{\mathrm{d}t} = y \cdot \left(1 - \frac{y}{3}\right)$$

   c.
   $$x'(t) = a \cdot x(t) \cdot e^{-b \cdot x(t)}$$

   answer:
   $$\frac{\mathrm{d}x}{\mathrm{d}t} = a \cdot x \cdot e^{-b \cdot x}$$

2. Consider the following model:
   $$\frac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 - \frac{x}{10}\right)$$
   What is the instantaneous change in x at the following points?

   a. $x(t) = 10/2$
   answer: Filling in $x = 10/2$ gives the instantaneous change at this point: $(5/2)$

   b. $x(t) = 10$
   answer: 0

   c. $x(t) = 20$
   answer: -20

3. From the rules for derivatives it follows that if $z(t) = a \cdot x(t)$ then $z'(t) = a \cdot x'(t)$.

   a. Write this expression in terms of $\frac{\mathrm{d}z}{\mathrm{d}t}$ and $\frac{\mathrm{d}x}{\mathrm{d}t}$
   answer:
   $$\frac{\mathrm{d}z}{\mathrm{d}t} = a \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$$

b. If it is given that $\frac{dx}{dt} = 2 \cdot x + 3$, what is the differential equation for $z = 2 \cdot x$?
answer:

$$\frac{dz}{dt} = 2\,(2 \cdot x + 3) \Rightarrow$$
$$\frac{dz}{dt} = 2 \cdot z + 6$$

section 12.2

1. Consider the solution in (12.11)

a. Compute the derivative of the expression for $x(t)$ in this equation.
answer:

$$x'\,(t) = x\,(0) \cdot a \cdot e^{at}$$

b. Show that the outcome is equal to $a \cdot x(t)$
answer:

$$\begin{aligned} x(t) &= x\,(0) \cdot e^{at}, \\ x'\,(t) &= x\,(0) \cdot a \cdot e^{at} \\ &= a \cdot \left(x\,(0) \cdot e^{at}\right) \end{aligned}$$

2. Use R, Excel, or another device to compute the approximate solution of the differential equation $\frac{dx}{dt} = 2 \cdot x$, with $x(0) = 1$, by means of Euler integration. Compare the outcomes for $h = 0.1$ and $h = 0.01$ up to $t = 1$ with the real solution $x(t) = e^{2t}$.

3. Derive an approximation for the differential equation $\frac{dx}{dt} = 2 \cdot x + 1$ for small time steps h, using the same method as in (12.8).
answer:

$$\begin{aligned} \frac{x\,(t+h) - x\,(t)}{h} &\approx 2 \cdot x\,(t) + 1 \\ x\,(t+h) - x\,(t) &\approx 2h \cdot x\,(t) + h \\ x\,(t+h) &\approx x\,(t) + 2h \cdot x\,(t) + h \end{aligned}$$

12.4.1

1. What are the equilibrium points of the following model:

$$\frac{dx}{dt} = a \cdot x - \frac{b \cdot x}{1 + x}$$

answer: $\hat{x}_1 = 0,\ \hat{x}_2 = \frac{b-a}{a}$

2. The so-called Allee model for population dynamics is:

$$\frac{dx}{dt} = r \cdot x \cdot \left(\frac{x}{M} - 1\right) \cdot \left(1 - \frac{x}{K}\right), \text{with} x \geq 0, r > 0, M > 0, K > M \qquad (14.6)$$

r and K are the intrinsic growth rate and the carrying capacity (just as in the logistic model), and M is called the minimum viable population size. What are the equilibria of this model?
answer: $\hat{x}_1 = 0,\ \hat{x}_2 = M,\ \hat{x}_3 = K$

3. The following model adds proportional harvesting at a rate h to the logistic model:

$$\frac{dx}{dt} = r \cdot x \cdot \left(1 - \frac{x}{K}\right) - h \cdot x,\ \text{with } x \geq 0,\ r > 0,\ K > 0,\ h > 0 \qquad (14.7)$$

a. What is the equilibrium equation for this model?
answer: $r \cdot \hat{x} \cdot \left(1 - \frac{\hat{x}}{K}\right) - h \cdot \hat{x} = 0$

b. This model has at most two equilibria. What are their values?
answer: $\hat{x}_1 = 0,\ \hat{x}_2 = \frac{K \cdot (r-h)}{r}$

c. For which values of h does the model have a positive equilibrium?
answer: $h < r$

section 12.4.2

1. Consider the quadratic model:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2 \cdot x^2 \tag{14.8}$$

a. This model has one equilibrium. What is its value?
answer: $\hat{x} = 0$

b. What is the instantaneous change in x in this model at the points $x(t) = -1$ and $x(t) = 1$?
answer: When $x(t) = -1$ it is 2, when $x(t) = 1$ it is 2

c. When the instantaneous change in x is positive, the value of $x(t)$ increases in time, when it is negative, it decreases. What does this model predict concerning the value of $x(t)$ if you start at a value of $x(0) = -1$?
answer: $x(t)$ increases

d. What does this model predict concerning the value of $x(t)$ if you start at a value of $x(0) = +1$?
answer: $x(t)$ increases

e. For which values of $x(0)$ does $x(t)$ move towards the equilibrium and for which values does it move away?
answer: When $x(0) < 0$ it moves towards the equilibrium, when $x(0) > 0$ it moves away.

f. What type of equilibrium is $\hat{x}$?
answer: It is a saddle point attracting from the left and repelling from the right.

2. Consider the quadratic model:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -2 \cdot x^2$$

a. This model has one equilibrium. What is its value?
answer: one: $\hat{x} = 0$

b. What is the instantaneous change in x in this model at the points $x = -1$ and $x = 1$?
answer: When $x = -1$ it is -2, when $x = 1$ it is -2

c. What does this model predict concerning the value of $x(t)$ if you start at a value of $x(0) = -1$?
answer: $x(t)$ decreases

d. What does this model predict concerning the value of $x(t)$ if you start at a value of $x(0) = +1$?
answer: $x(t)$ decreases

e. What type of equilibrium is $\hat{x}$?
answer: It is a saddle point attracting from the right and repelling from the lefy.

section 12.4.3

1. Consider the following model:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x \cdot \left(\frac{x}{3} - 1\right) \cdot \left(1 - \frac{x}{20}\right) \tag{14.9}$$

a. What is the function $f(x)$ in this case?
answer: $f(x) = x \cdot \left(\frac{x}{3} - 1\right) \cdot \left(1 - \frac{x}{20}\right)$

b. What is the equilibrium equation?
answer: $x \cdot \left(\frac{x}{3} - 1\right) \cdot \left(1 - \frac{x}{20}\right) = 0$
c. What are the equilibria of this model?
answer: $0, 3, 20$
d. What is the derivative $f'(x)$?
answer:

$$\left(1 - \frac{x}{20}\right)\left(-1 + \frac{x}{3}\right) + \frac{1}{3}\left(1 - \frac{x}{20}\right)x - \frac{1}{20}\left(-1 + \frac{x}{3}\right)x$$
$$\frac{1}{60}\left(-60 + 46x - 3x^2\right)$$

e. Use the method presented in this section to determine which of the equilibria are locally stable, and which are unstable.
answer:
$f'(0) = -1 < 0$ so the equilibrium at 0 is stable.
$f'(3) = \frac{17}{20} > 0$, so the equilibrium at 3 is unstable.
$f'(20) = -\frac{17}{3} < 0$, so the equilibrium at 20 is stable.

2. Consider the quadratic model:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a \cdot x^2 - h \cdot x \tag{14.10}$$

a. What is the function $f(x)$ in this case?
answer: $f(x) = a \cdot x^2 - h \cdot x$
b. What is the equilibrium equation?
answer: $a \cdot x^2 - h \cdot x = 0$
c. What are the equilibria of this model?
answer: 0 and h/a
d. What is the derivative $f'(x)$?
answer: $f'(x) = 2ax - h$
e. How does the stability of the equilibria depend on the values of a and h?
answer:
$f'(0) = -h$. The equilibrium at 0 is stable when $h > 0$ and unstable when $h < 0$.
$f'(h/a) = h$. The equilibrium at h/a is stable when $h < 0$ and unstable when $h > 0$.
f. When $h = 0$ there is a single equilibrium. What is its value?
answer: 0
g. What type of equilibrium is this?
answer: a saddle point

***section* 12.4.5**

1. Make a graph of $\frac{\mathrm{d}x}{\mathrm{d}t}$ versus x for the logistic model (Eq. (12.6)). What is the region of atraction of the carrying capacity equilibrium?
answer: The graph is a parabola opening down, with x-intercepts at 0 and K. The region of attraction of K consists of all $x > 0$.
2. Examine the effects of harvesting a logistic model, as in assignment 3. What happens to the graph of $\frac{\mathrm{d}x}{\mathrm{d}t}$ versus x as h increases? What does this imply for the population?
answer: The second equilibrium point becomes smaller as h increases. The population size becomes smaller.
3. Make a graph of the following Allee model with harvesting:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x \cdot \left(\frac{x}{3} - 1\right) \cdot \left(1 - \frac{x}{10}\right) - h \cdot x, \text{ with } x \geq 0, h > 0 \tag{14.11}$$

Examine what happens when h increases, and discuss what this implies for the population.
answer: The two positive equilibria move closer together, implying that the population size shrinks, and the region of attraction of the ’carrying capacity equilibrium’ becomes smaller. At some point, the two equilibria ‘collapse’ into a single saddle point. As h increases further, only the 0 equilibrium is left, and it is stable. The population will then go extinct.

Chapter 13

Deterministic, continuous time, multivariate models

section 13.1.1

1. Write the following systems of equations in matrix notation:

   a.

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 + 3x_2\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 2x_1 + x_2\end{aligned} \tag{14.12}$$

   answer:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \cdot \mathbf{x}$$

   b.

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -x_1 + x_2\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1 - x_2\end{aligned} \tag{14.13}$$

   answer:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \mathbf{x}$$

   c.

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 2x_2\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1 - x_2\end{aligned} \tag{14.14}$$

   answer:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \cdot \mathbf{x}$$

13

1. Show that the model in (13.3) has only one equilibrium point, at $(0,0)$.
   answer: The first equation gives $2 \cdot \hat{x}_1 = 3 \cdot \hat{x}_2 \Rightarrow \hat{x}_1 = \frac{3}{2} \cdot \hat{x}_2$. Substituting this in the second equation we get $2 \cdot \frac{3}{2} \cdot \hat{x}_2 + 4 \cdot \hat{x}_2 = 0$. The only value that satisfies this is $\hat{x}_2 = 0$. This implies that $\hat{x}_1 = 0$.
2. Consider the following model:

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 + 3x_2\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -2x_1 - 6x_2\end{aligned}$$

a. Give the equilibrium equations.
answer: The equilibrium equations are:

$$\hat{x}_1 + 3\hat{x}_2 = 0$$
$$-2\hat{x}_1 - 6\hat{x}_2 = 0$$

b. How many equilibria are there?
answer: The first equation gives $\hat{x}_1 = -3\hat{x}_2$. Every combination of $\hat{x}_1$ and $\hat{x}_2$ values that satisfies this equation, also satisfies the second one. So there are infinitely many equilibria.

c. What does the set of equilibria look like in a a graph with x_2 on the vertical axis, and x_1 on the horizontal axis?
answer: The equilibria are all points on the line with coordinates $y = -1/3x$.

3. Consider the general model in Eq. (13.5)

a. Write down the equilibrium equations for this model.
answer:

$$a \cdot \hat{x}_1 + b \cdot \hat{x}_2 = 0$$
$$c \cdot \hat{x}_1 + d \cdot \hat{x}_2 = 0$$

b. Show that when $a = 0$ and all other parameters are nonzero that $(0, 0)$ is the only equilibrium point.
answer: In this case the first equilibrium equation becomes: $b \cdot \hat{x}_2 = 0$, so $\hat{x}_2 = 0$. Substitution in the second equation gives: $c \cdot \hat{x}_1 = 0$, which implies $\hat{x}_1 = 0$.

c. What happens when $a = 0$ and $b = 0$?
answer: Then all points with $\hat{x}_2 = 0$ are equilibria.

d. Show that there are infinitely many equilibria when $a \cdot d - b \cdot c = 0$.
answer: The first equilibrium equation gives $\hat{x}_1 = -\frac{b}{a}\hat{x}_2$, substitution in the second equation gives:

$$c \cdot \left(-\frac{b}{a}\hat{x}_2\right) = -d \cdot \hat{x}_2$$

so $\hat{x}_2 = 0$ or $c \cdot \left(-\frac{b}{a}\right) = -d \Rightarrow bc = ad$. If the latter equality holds, all values of x_2 suffice.

e. What does the set of equilibria look like in a a graph with x_2 on the vertical axis, and x_1 on the horizontal axis?
answer: The equilibria are all points on the line with coordinates $y = -a/b \cdot x$.

section 13.1.3

1. Use Excel, R, or a graphing calculator to draw graphs of the functions $x_1(t)$ and $x_2(t)$ in (13.17) and their approximations in (13.18).
2. Use Excel, R, or a graphing calculator to draw graphs of the functions $x_1(t)$ and $x_2(t)$ in (13.24).
3. Consider the solution in (13.17).

a. Calculate $\frac{dx_1}{dt}$ by differentiating the right hand side of the first equation.
answer:

$$\frac{dx_1}{dt} = e^{-2t} + 3e^{2t}$$

b. Show that the outcome is equal to $x_1(t) + 3 \cdot x_2(t)$.
answer:

$$\begin{aligned} x_1(t) + 3x_2(t) &= -\frac{1}{2}e^{-2t} + \frac{3}{2}e^{2t} + \frac{3}{2}e^{-2t} + \frac{3}{2}e^{2t} \\ &= e^{-2t} + 3e^{2t} \end{aligned}$$

c. Calculate $\frac{dx_2}{dt}$ by differentiating the right hand side of the second equation.
answer:

$$\frac{dx_2}{dt} = -e^{-2t} + e^{2t}$$

d. Show that the outcome is equal to $x_1(t) - x_2(t)$.
answer:

$$\begin{aligned} x_1(t) - x_2(t) &= -\frac{1}{2}e^{-2t} + \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t} - \frac{1}{2}e^{2t} \\ &= -e^{-2t} + e^{2t} \end{aligned}$$

e. What can you conclude from these results?
answer: The solution in (13.17) indeed satisfies the differential equations in (13.14).

4. Consider the solution in (13.24).

a. Calculate $\frac{dx_1}{dt}$ by differentiating the right hand side of the first equation.
answer:

$$\begin{aligned} \frac{dx_1}{dt} &= -e^{-t} \cdot \left(\cos(2t) - \frac{1}{2}\sin(2t)\right) + e^{-t}\left(-2\sin(2t) - \cos(2t)\right) \\ &= e^{-t} \cdot \left(-2\cos(2t) - \frac{3}{2}\sin(2t)\right) \end{aligned}$$

b. Show that the outcome is equal to $-x_1(t) - x_2(t)$.
answer:

$$\begin{aligned} -x_1(t) - x_2(t) &= -e^{-t}\left(\cos(2t) - \frac{1}{2}\sin(2t)\right) - e^{-t}\left(2\sin(2t) + \cos(2t)\right) \\ &= e^{-t}\left(-2\cos(2t) - \frac{3}{2}\sin(2t)\right) \end{aligned}$$

c. Calculate $\frac{dx_2}{dt}$ by differentiating the right hand side of the second equation.
answer:

$$\begin{aligned} \frac{dx_2}{dt} &= -e^{-t}\left(2\sin(2t) + \cos(2t)\right) + e^{-t}\left(4\cos(2t) - 2\sin(2t)\right) \\ &= e^{-t}\left(-4\sin(2t) + 3\cos(2t)\right) \end{aligned}$$

d. Show that the outcome is equal to $4x_1(t) - x_2(t)$.
answer:

$$\begin{aligned} 4x_1(t) - x_2(t) &= 4e^{-t}\left(\cos(2t) - \frac{1}{2}\sin(2t)\right) - e^{-t}\left(2\sin(2t) + \cos(2t)\right) \\ &= e^{-t}\left(-4\sin(2t) + 3\cos(2t)\right) \end{aligned}$$

e. What can you conclude from these results?
answer: The solution in (13.24) indeed satisfies the differential equations in (13.21).

section 13.1.4

1. Determine the type of the equilibrium in each of the following models, based on the trace and determinant of the matrix **A**:

a.

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + 3x_2 \\ \frac{dx_2}{dt} &= x_1 + 2x_2 \end{aligned}$$

answer: $T = 4, D = 1$ The equilibrium is an unstable node.

b.
$$\frac{dx_1}{dt} = 3x_2$$
$$\frac{dx_2}{dt} = x_1$$
answer: $T = 0, D = -3$ The equilibrium is a saddle point.

c.
$$\frac{dx_1}{dt} = -x_1 + -2x_2$$
$$\frac{dx_2}{dt} = x_1 - x_2$$
answer: $T = -2, D = 3$ The equilibrium is a stable focus.

2. Determine how the stability of the equilibrium in each of the following models depends on p:

a.
$$\frac{dx_1}{dt} = 2p \cdot x_1 + 3x_2$$
$$\frac{dx_2}{dt} = x_1 + 2x_2$$
answer: $T = 2 + 2p$. The equilibrium would be stable if $T < 0$ and $D > 0$, so if $p < -1$ and $p > 3/4$. This means that it is never stable. The equilibrium is neutrally stable if $T = 0$ and $D > 0$, which also is not possible. If $D = 0$ there are infinitely many, neutrally stable equilibria. This happens when $p = 3/4$.

b.
$$\frac{dx_1}{dt} = x_1 - px_2$$
$$\frac{dx_2}{dt} = x_1$$
answer: $T = 1, D = p$ The equilibrium is unstable unless p=0, then there are infinitely many, neutrally stable equilibria.

3. Determine for which values of p the equilibrium of the following model is a stable focus:
$$\frac{dx_1}{dt} = -x_1 - px_2$$
$$\frac{dx_2}{dt} = x_1$$
answer: $T = -1, D = p$. The equilibrium is a stable focus if $D > \frac{1}{4}T^2$, so if $p > 1/4$.

13.1.5

1. Consider the following model:
$$\frac{dx_1}{dt} = 2x_1 + 3x_2 - 1$$
$$\frac{dx_2}{dt} = x_1 - 4x_2 + 3$$

a. What are the matrix **A**, and the vector **b** in this case?
answer:
$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

b. What are the equations for the equilibrium?
answer:
$$2\hat{x}_1 + 3\hat{x}_2 - 1 = 0$$
$$\hat{x}_1 - 4\hat{x}_2 + 3 = 0$$

c. What are the coordinates of the equilibrium?
answer: $\left(-\frac{5}{11}, \frac{7}{11}\right)$

d. What are the trace and determinant of **A**?
answer: $T = -2, D = -11$

e. What type of equilibrium is this?
answer: A saddle point.

2. Answer the same questions as in the previous exercise, for the model:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 4x_1 - x_2 + 1$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_1 + x_2$$

answer:

a.

$$\mathbf{A} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

b. equilibrium equations:

$$4\hat{x}_1 - \hat{x}_2 + 1 = 0$$
$$-\hat{x}_1 + \hat{x}_2 = 0$$

c. equilibrium: $\left(-\frac{1}{3}, -\frac{1}{3}\right)$

d. $T = 5, D = 6$.

e. This is an unstable node.

13.2

1. Identify the functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ for each of the following models:

a.

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 2{x_1}^2 + 3x_1 \cdot (x_2 - 1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1$$

answer:

$$f_1(x_1, x_2) = 2{x_1}^2 + 3x_1 \cdot (x_2 - 1)$$
$$f_2(x_1, x_2) = x_1$$

b.

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{x_1}{1 + 2x_1} - x_2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 2x_1 + x_2$$

answer:

$$f_1(x_1, x_2) = \frac{x_1}{1 + 2x_1} - x_2$$
$$f_2(x_1, x_2) = 2x_1 + x_2$$

Exercises and assignments

1. What are the equilibrium points of the following models?:

   a.
$$\begin{aligned}\frac{dx_1}{dt} &= x_1 - 3x_1x_2\\ \frac{dx_2}{dt} &= 2x_1x_2 - x_2\end{aligned} \tag{14.15}$$

   answer: The equilibrium equations are:

$$\begin{aligned}\hat{x}_1 - 3\hat{x}_1\hat{x}_2 &= 0\\ 2\hat{x}_1\hat{x}_2 - \hat{x}_2 &= 0\end{aligned}$$

   The first equation gives $\hat{x}_1 = 0$, or $\hat{x}_2 = \frac{1}{3}$. Filling in $\hat{x}_1 = 0$ in the second equation gives $\hat{x}_2 = 0$, so $(0,0)$ is one equilibrium point. Substituting $\hat{x}_2 = \frac{1}{3}$ in the second equation gives $\hat{x}_1 = 1/2$, so the other equilibrium point is $(1/2, 1/3)$.

   b.
$$\begin{aligned}\frac{dx_1}{dt} &= 4x_1\left(1 - \frac{x_1}{2}\right) - 3x_1x_2\\ \frac{dx_2}{dt} &= 3x_1x_2 - 2x_2\end{aligned} \tag{14.16}$$

   answer: The equilibrium equations are:

$$\begin{aligned}4\hat{x}_1\left(1 - \frac{\hat{x}_1}{2}\right) - 3\hat{x}_1\hat{x}_2 &= 0\\ 3\hat{x}_1\hat{x}_2 - 2\hat{x}_2 &= 0\end{aligned}$$

   The second equation gives $\hat{x}_2 = 0$, or $\hat{x}_1 = \frac{2}{3}$. Substituting $\hat{x}_2 = 0$ in the first equation gives $\hat{x}_1 = 0$ or $\hat{x}_1 = 2$, this gives two equilibrium points: $(0,0)$ and $(2,0)$. Substituting $\hat{x}_1 = \frac{2}{3}$ in the first equation gives $\hat{x}_2 = 8/9$, so the third equilibrium is $(2/3, 8/9)$

section 13.2.2

1. What are the trace and the determinant of the Jacobian matrix for the other equilibrium point, $(1,3/2)$ in the example? What type of equilibrium is this?
   answer: In this case:
$$\mathbf{F}'(\hat{\mathbf{x}}) = \begin{pmatrix} 0 & -2 \\ \frac{3}{2} & 0 \end{pmatrix}$$
   with trace $T = 0$ and determinant $D = 3$. The equilibrium is a center.

2. Consider the model in (13.52)

   a. What is the Jacobian matrix for this model?
   answer:
$$\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} 1 - 3x_2 & -3x_1 \\ 2x_2 & 2x_1 - 1 \end{pmatrix}$$

   b. What is the Jacobian, evaluated at the equilibrium $(0,0)$?
   answer:
$$\mathbf{F}'(\hat{\mathbf{x}}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

c. What type of equilibrium is this?
answer: $T = 0, D = -1$. The equilibrium is a saddle point.

d. What type of equilbrium is the point $(1/2, 1/3)$?
answer: The Jacobian evaluated at the equilibrium $(1/2, 1/3)$ is:

$$\mathbf{F}'(\hat{\mathbf{x}}) = \begin{pmatrix} 0 & -3/2 \\ 2/3 & 0 \end{pmatrix}$$

$T = 0, D = 1$. The equilibrium is a center.

3. Consider the model in (13.53).

a. What is the Jacobian matrix for this model?
answer:

$$\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} 4 - 4x_1 - 3x_2 & -3x_1 \\ 3x_2 & 3x_1 - 2 \end{pmatrix}$$

b. What type of equilibrium is $(0, 0)$?
answer: The Jacobian matrix, evaluated at this point is:

$$\mathbf{F}'(\hat{\mathbf{x}}) = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$$

with $T = 2, D = -8$.This is a saddle point.

c. What type of equilibrium is $(2, 0)$?
answer: The Jacobian matrix, evaluated at this point is:

$$\mathbf{F}'(\hat{\mathbf{x}}) = \begin{pmatrix} -4 & -6 \\ 0 & 4 \end{pmatrix}$$

with $T = 0, D = -16$. This is a saddle point.

d. What type of equilibrium is $(2/3, 8/9)$?
answer: The Jacobian matrix, evaluated at this point is:

$$\mathbf{F}'(\hat{\mathbf{x}}) = \begin{pmatrix} -\frac{4}{3} & -2 \\ \frac{8}{3} & 0 \end{pmatrix}$$

with $T = -\frac{4}{3}, D = 16$. This is a stable focus.

Part III

Mathematical foundation

15
Numbers and sets

15.1 Number systems

The most basic mathematical entities are numbers. There are different sets of numbers. The most important ones, which we will briefly consider in this chapter, are indicated with special symbols. The number systems evolved in the course of human history. Initially, numbers were used for counting, then people invented the number zero, and negative numbers were needed. As people started to make more intricate calculations, and mathematics evolved, further number sets were invented (or we could also say, discovered).

The set of *natural numbers* is denoted by $\mathbb{N}$. These are the counting numbers, $1, 2, 3$, etc.. The set of *whole numbers*, indicated by $\mathbb{N}_0$ is this set including the number 0. With this set we can indicate quantities of separate entities, such as, for instance, numbers of individuals. If natural (or whole) numbers are added to or multiplied with each other, the result is again a natural number. If they are subtracted from each other, however, the result might be negative, and lie outside the original set. Similarly. division of natural numbers usually produces a non-natural number (for instance 1/3).

Mathematicians would say that these sets are *closed* with respect to addition and multiplication.

The set of *integers*, denoted by $\mathbb{Z}$ includes the counting numbers, zero, and the negative numbers. This set is closed with respect to addition, multiplication, and subtraction. This set of numbers may be used, for instance, to indicate debts or shortages.

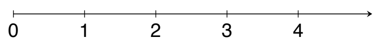


Figure 15.1: The set $\mathbb{N}_0$ on a number line.

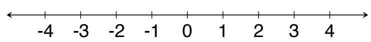


Figure 15.2: Some numbers of the set $\mathbb{Z}$ on a number line.

The sets that we have considered up to now do not include numbers that result from divisions. The set of *rational numbers*, indicated by $\mathbb{Q}$, consists of the set $\mathbb{Z}$ together with the non-integer numbers that result from divisions, such as for instance $2/3$, or $5/4$. This set is closed with respect to the operations addition, subtraction, multiplication and division. It may be used for instance, to calculate interest on a loan or a bank account, or change the scale of figures, or structures.

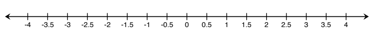


Figure 15.3: Some numbers of the set $\mathbb{Q}$ on a number line.

It is rumoured that the Pythagoreans were very upset when they discovered that there were numbers that cannot be expressed as a fraction of integers. These numbers are called *irrational* numbers. Famous examples are the number π, and Euler's number e. Another example is the square root of 2. The set of real numbers extends $\mathbb{Q}$, by including the irrational numbers. The resulting set is denoted by $\mathbb{R}$. This set is closed with respect to addition, subtraction, multiplication and division, and also to raising a number to a whole power, and raising positive numbers to fractional powers. Real numbers are needed when calculating sides of triangles, or the circumference, surface, or volume of spherical figures. Such calculations occur for instance in architectural or agricultural applications.

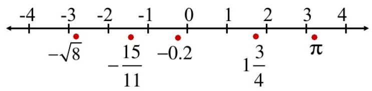


Figure 15.4: Some numbers of the set $\mathbb{R}$ on a number line.

16
Summations and products

A *sequence* is an ordered list of numbers. For instance: $1, 4, 8, 9$ is a sequence of four numbers. The numbers in the sequence are called its *elements*. The individual elements of a sequence may be indicated by a symbol with a subscript, for instance in the example, $x_1 = 1, x_2 = 4, x_3 = 8, x_4 = 9$. In this example, the numbers are ordered according to their magnitude. This is not always the case. In statistical applications, for instance, the elements may simply be represented in the order in which they were measured, for instance the following sequence may represent the ages of four successive arrivals at a concert: $23, 17, 18, 17, 20$. In this case, $x_1 = 23, x_2 = 17, x_3 = 18, x_4 = 17, x_5 = 20$. The latter example also illustrates that some elements in the list may have identical values (in this example $x_2 = x_4$).

16.1 Sums of sequences

The mathematical notation for a sum of a sequence is as follows:

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n \tag{16.1}$$

where n is the total number of elements. In the first example, $n = 4$, and

$$\sum_{i=1}^{n} x_i = 1 + 4 + 8 + 9 = 22 \tag{16.2}$$

In the second example, $n = 5$, and

$$\sum_{i=1}^{n} x_i = 23 + 17 + 18 + 17 + 20 = 95 \tag{16.3}$$

The average of a sequence is often denoted by: $\bar{x}$, and, using the summation sign it can be written as:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{16.4}$$

From the standard rules for calculations of sums it follows that for any constant a:

$$a \cdot \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a \cdot x_i \tag{16.5}$$

For example:

$$\begin{aligned} &2 \cdot (1+4+8+9) = 2 \cdot 22 \\ &-4 \cdot (23+17+18+17+20) = -4 \cdot 95 \end{aligned} \tag{16.6}$$

Furthermore, for any two sums with the same number of elements we have:

$$\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} (x_i + y_i) \tag{16.7}$$

For example:

$$\begin{aligned} &(1+4+8+9) + (23+17+18+17) = \\ &(1+23) + (4+17) + (8+18) + (9+17) \end{aligned} \tag{16.8}$$

Combining the previous statements, we can formulate the more general rule:

$$a \cdot \sum_{i=1}^{n} x_i + b \cdot \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} (a \cdot x_i + b \cdot y_i) \tag{16.9}$$

Note that the summation of a constant results in n times that constant, formally:

$$\sum_{i=1}^{n} a = a + a + \ldots a = n \cdot a \tag{16.10}$$

Since the average, $\bar{x}$ is a constant, this gives for instance an alternative expression for the variance:

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 &= \frac{1}{n} \sum_{i=1}^{n} \left({x_i}^2 - 2x_i \cdot \bar{x} + \bar{x}^2 \right) \\ &= \frac{1}{n} \sum_{i=1}^{n} {x_i}^2 - \frac{1}{n} \sum_{i=1}^{n} 2x_i \cdot \bar{x} + \frac{1}{n} \sum_{i=1}^{n} \bar{x}^2 \\ &= \frac{1}{n} \sum_{i=1}^{n} {x_i}^2 - 2 \cdot \bar{x} \cdot \frac{1}{n} \sum_{i=1}^{n} x_i + \frac{1}{n} \cdot n \cdot \bar{x}^2 \\ &= \frac{1}{n} \sum_{i=1}^{n} {x_i}^2 - 2\bar{x}^2 + \bar{x}^2 \\ &= \frac{1}{n} \sum_{i=1}^{n} {x_i}^2 - \bar{x}^2 \end{aligned} \tag{16.11}$$

In most statistical software a slightly different expression for the variance is used, namely $\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$. This is a bit more accurate for small sample sizes.

Exercises and assignments

1. Consider the following sequence $3, 5, 5, 2, 6, 1, 4$.
   a. The individual elements are referred to as $x_i,\ i = 1, \ldots, 7$. What is the value of x_5?
   b. What is the length of the series, n?
   c. Calculate $\sum_{i=1}^{n} x_i$
   d. Calculate $\sum_{i=2}^{n-1} x_i$

e. What is the value of $\bar{x}$?

f. Calculate $\sum_{i=1}^{n} {x_i}^2$

g. What is the variance of the data?

2. Show that

$$\sum_{i=1}^{n} (a \cdot x_i)^2 = a^2 \cdot \sum_{i=1}^{n} {x_i}^2$$

Use the relation in (16.5).

3. Show that

$$\frac{1}{n} \sum_{i=1}^{n} (x_i - b) = \bar{x} - b$$

Use the relation in (16.10).

16.2 Products of sequences

The mathematical notation for a product of a sequence is:

$$\prod_{i=1}^{n} x_i = x_1 \cdot x_2 \ldots \cdot x_n \qquad (16.12)$$

For instance, the product of the sequence $1, 4, 8, 9$ is:

$$\prod_{i=1}^{n} x_i = 1 \cdot 4 \cdot 8 \cdot 9 = 288 \qquad (16.13)$$

From the standard rules for calculations of products it follows that for any constant a:

$$\prod_{i=1}^{n} a \cdot x_i = a \cdot \prod_{i=1}^{n} x_i \qquad (16.14)$$

and for any two sequences of equal length we have:

$$\left(\prod_{i=1}^{n} x_i\right) \cdot \left(\prod_{i=1}^{n} y_i\right) = \prod_{i=1}^{n} x_i \cdot y_i \qquad (16.15)$$

Exercises and assignments

1. The individual elements of the following sequences are referred to as $x_i, i = 1, \ldots n$, where n is the sequence length. For each of the sequences calculate $\prod_{i=1}^{n} x_i$.

a. $1, 4, 1, 5, 10$

b. $31, 21, 7, 0, 3, 16, 10$

c. $2, 2, 2, 2, 1,$

17 Properties of elementary functions

17.1 Polynomials

Polynomials are functions of the form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_n x^n,\ a_n \neq 0, \tag{17.1}$$

where n is a whole number, which is called the *degree* of a polynomial. The parameters a_0 up to a_n may have any value (provided a_n is not 0, otherwise the polynomial has a lower degree). Here are some examples:

$$f(x) = 3 + x + 3x^2 + 5x^5 \tag{17.2}$$

This is a polynomial of degree 5, with $a_0 = 3, a_1 = 1, a_2 = 3, a_4 = 0,$ and $a_5 = 5$.

$$f(x) = x^6 \tag{17.3}$$

This is a polynomial of degree 6, with a_0 to a_5 equal to 0, and $a_6 = 1$.

Polynomials are used in many contexts. For instance, to interpolate between successive values in tables, as functions that are fitted through data points in statistics, and to approximate complex, nonlinear functions in modelling.

17.1.1 Linear functions

A zero-degree polynomial is simply a constant:

$$f(x) = a_0 \tag{17.4}$$

An example of a zero-degree polynomial is $f(x) = 2$. The graph of such a function is a horizontal line, at the value $y = a_0$.

A first-degree polynomial is a non-constant linear function:

$$f(x) = a_0 + a_1 x,\ a_1 \neq 0 \tag{17.5}$$

For example: $f(x) = 3 + 2x$ is a first-degree polynomial. The graph of a first-degree polynomial function is a straight line, with slope equal to a_1 and y-intercept a_0. If the slope is positive, $a_1 > 0$, the function is increasing in x. If it is negative $a_1 > 0$, the function decreases in x. The function intersects the vertical axis at the point

$y = a_0$. The point of intersection with the horizontal axis, the x-intercept, equals $-\frac{a_0}{a_1}$. For instance, a graph of $f(x) = 3 + 2x$ is a straight line, with a positive slope that has a tangent of $a_1 = 2$. It has a y-intercept at $a_0 = 3$, and its x-intercept is $-\frac{a_0}{a_1} = -3/2$.

There are four different possible types of graphs, depending on the combinations of signs of a_0 and a_1. These are shown in Fig. 17.1.

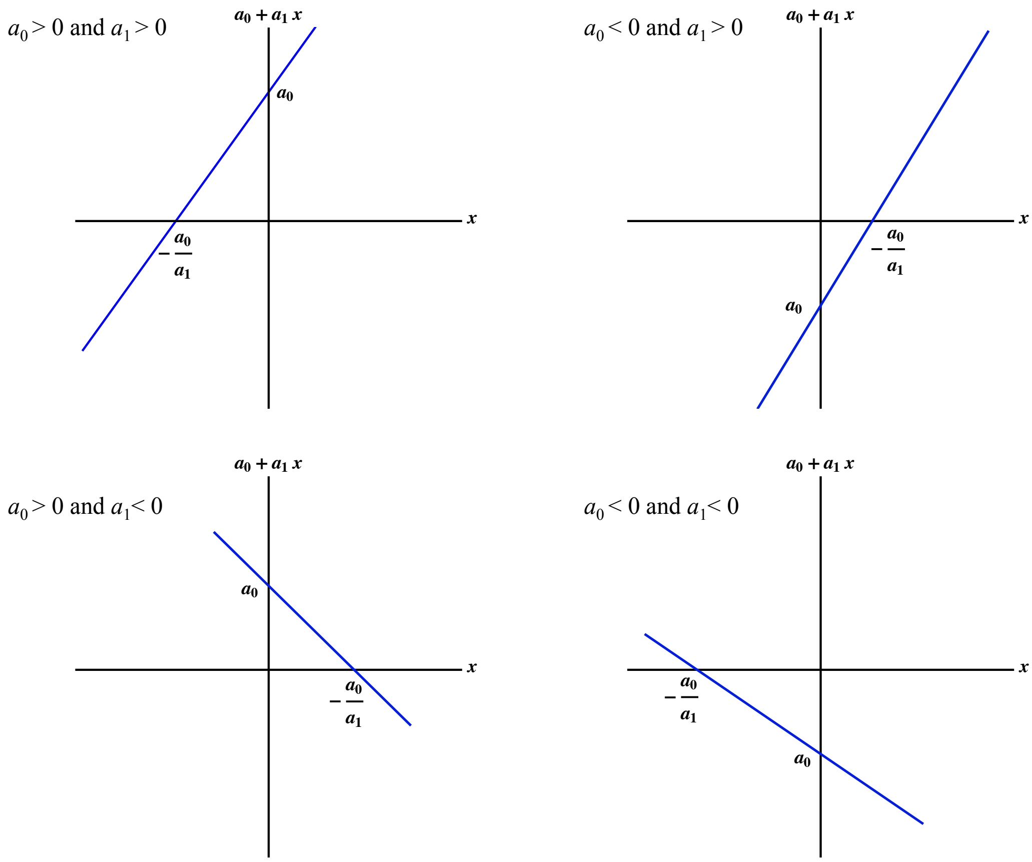


Figure 17.1: The different types of first-degree polynomials.

Exercises and assignments

1. Sketch the graphs of the following linear functions. Do this by calculating their intercepts and taking into account whether their slope is positive or negative. Indicate the values of the intercepts on the axes, and no scale, as was done in Fig. 17.1.

   a. $f(x) = 3 + 2x$
   b. $f(x) = -3 + 2x$
   c. $f(x) = 4 - 3x$
   d. $f(x) = -4 - 3x$

2. Calculate the x- and y-intercept for the following linear functions.

a. $f(x) = 4 + x$

b. $f(x) = -5 + 3x$

c. $f(x) = 1 - 6x$

d. $f(x) = -3 - 7x$

3. Sketch the graphs of each pair of the following basic functions in one figure.

a. $f(x) = -5 + 4x,\ g(x) = 2 + 4x$

b. $f(x) = 4 + 2x,\ g(x) = 4 + x$

4. For which parameter combinations do the following functions have a positive x-intercept?

a. $f(x) = \alpha \cdot x + 4$

b. $f(x) = 2 \cdot x - \beta$

c. $f(x) = \rho \cdot x + \gamma$

5. For which parameter combinations do the following functions have a positive y-intercept?

a. $f(x) = \alpha \cdot x + 4$

b. $f(x) = 2 \cdot x + \beta$

c. $f(x) = \gamma + \eta \cdot x$

6. For which parameter combinations do the following pairs of functions have the same slope?

a. $f(x) = \alpha \cdot x + 4,\ g(x) = 2x - 5 + x$

b. $f(x) = \alpha \cdot x + 2,\ g(x) = \beta \cdot x - 4 + 3\alpha$

c. $f(x) = \alpha \cdot x - \beta \cdot x + 1\ g(x) = 3x - 2$

17.1.2 *Quadratic functions*

A second-degree polynomial is a quadratic (also called *parabolic*) function:

$$f\,(x) = a_0 + a_1 x + a_2 x^2, a_2 \neq 0 \tag{17.6}$$

Some examples of quadratic functions are:

$$\begin{aligned} h(x) &= 1 + x^2 + 3x \\ g(x) &= 4x - 2x^2 \end{aligned} \tag{17.7}$$

The graph of such a function is called a *parabola*. This is a (mirror-)symmetrical shape. Its axis of symmetry is the vertical line $x = -\frac{a_1}{2a_2}$. This is the x-value at which the function has an extremum (i.e. a maximum or minimum). For instance, the axis of symmetry of $h(x)$ is at $-\frac{3}{2\cdot 1} = -\frac{3}{2}$ and that of $g(x)$ at $-\frac{4}{2\cdot(-2)} = 1$.

The function is u-shaped (opening to the top) if $a_2 > 0$, in this case the extremum is a minimum. If $a_2 < 0$, the function is opening to the bottom, and it has a maximum. The function $h(x)$ has a minimum at $x = -\frac{3}{2}$ and $g(x)$ has a maximum at $x = -1$.

The y-intercept, the point at which it intersects with the vertical axis, is found by substituting $x = 0$ in equation (17.6), and equals a_0. The y-intercept of $h(x)$ is 1 and that of $g(x)$ equals 0.

Whether or not the function intersects the x-axis is determined by its discriminant:

$$D = {a_1}^2 - 4 \cdot a_2 \cdot a_0 \tag{17.8}$$

If $D < 0$ there are no x-intercepts. If $D = 0$ the x-axis is tangent to the function at its extremum. If $D > 0$ the function has two x-intercepts, at the values:

$$x = \frac{-a_1 \pm \sqrt{D}}{2a_2} \tag{17.9}$$

The discriminant for $h(x)$ is $3^2 - 4 \cdot 1 \cdot 1 = 5$, so this function has two x-intercepts. The discriminant for $g(x)$ is $16 + 4 \cdot 2 \cdot 0 = 16$, so this function also has two x-intercepts.
There are six different possible shapes of graphs, depending on the parameter combinations. These are shown in Fig. 17.2

Exercises and assignments

1. Indicate whether the following functions have a maximum or a minimum, give the x-value of the extremum, and calculate their intercepts for both axes.

   a. $f(x) = x^2 - 2x + 3$

   b. $f(x) = 4 + 4x - 2x^2$

   c. $f(x) = 6 - 2x + 9x^2$

2. Indicate whether the following functions have a maximum or a minimum, give the x-value of the extremum, and calculate their y-intercept.

   a. $f(x) = \alpha \cdot x^2 - 3x + 1$, with $\alpha > 0$

   b. $f(x) = 2 + \beta \cdot x - 4x^2$

   c. $f(x) = 3x^2 - 2x + \gamma$

3. For which parameter combinations do the following functions have a maximum?

   a. $f(x) = 3x^2 + \alpha \cdot x - 1$

   b. $f(x) = 3 - 2x - \beta \cdot x^2$

   c. $f(x) = (1 - \gamma)x^2 - x + 4$

4. For which parameter combinations do the following functions have two x-intercepts?

   a. $f(x) = \alpha \cdot x^2 - 3x + 1$

   b. $f(x) = x^2 + \gamma \cdot x - 3$

   c. $f(x) = 3x^2 - 4x + \beta$

17.1.3 *Third-degree polynomials*

A third-degree polynomial is defined by:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3, a_3 \neq 0 \tag{17.10}$$

An example of a third degree polynomial function is:

$$g(x) = 1 + 3x + 2x^2 + x^3 \tag{17.11}$$

The graph of this function can have different shapes. It either has two extrema: one maximum and one minimum, and an inflection point in-between, or it is a monotonic function (increasing or decreasing for all x-values) with no extrema, and one inflection point (see examples in Fig. 17.3)The exact shape may be determined by taking the derivative of the function, which corresponds to a second-order polynomial:

See chapter 19.

$$f'(x) = a_1 + 2 \cdot a_2 x + 3 \cdot a_3 x^2 \tag{17.12}$$

For instance, for the example in (17.11):

$$g'(x) = 3 + 4x + 3x^2 \tag{17.13}$$

If the equation $f'(x) = 0$ has two different, real-valued roots, the function $f(x)$ has two extrema. If the equation has one root, the function has an inflection point with a tangent of zero. If the equation has zero roots, the function has no extrema an inflection point with a tangent that is non-zero. For the function $g'(x)$ in (17.11), the discriminant is:

$$D = 4^2 - 4 \cdot 3 \cdot 3 < 0 \tag{17.14}$$

So the graph of this function has no extrema, just an inflection point.

The x-value of the inflection point is found by equating the second derivative to zero:

$$\begin{aligned} f''(x) = 2 \cdot a_2 + 6 \cdot a_3 x &= 0 \\ a_2 + 3 \cdot a_3 x &= 0 \\ x &= -\frac{a_2}{3 \cdot a_3} \end{aligned} \tag{17.15}$$

For instance, the inflection point of $g(x)$ lies at $x = -\frac{2}{3}$. By filling in this value in the derivative we find the tangent at the inflection point:

$$g'(-2/3) = 3 + 4 \cdot (-2/3) + 3 \cdot (-2/3)^2 = 5/3 \tag{17.16}$$

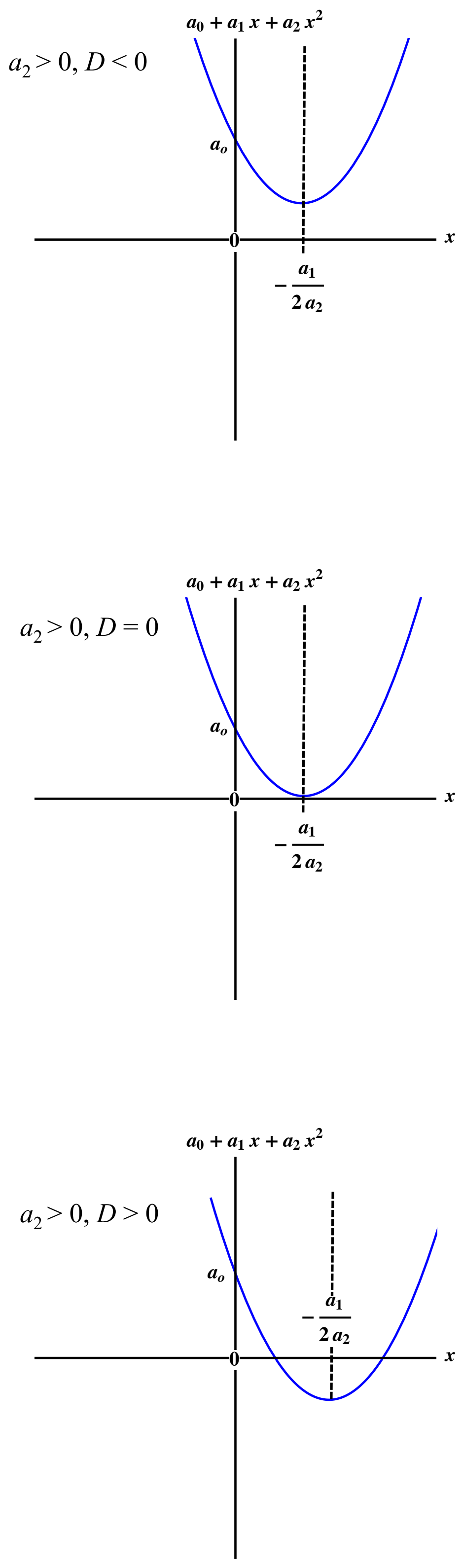


Figure 17.2: The different types of second-degree polynomials.

The graph of a third order polynomial function initially increases if $a_3 > 0$ and it decreases initially if $a_3 < 0$. The intercept with the vertical axis occurs at $y = a_0$ Examples of the different possible shapes that the graph may have are given in Fig. 17.3.

For the example of (17.11), $a_3 = 1 > 0$, so the function initially increases. Its y-intercept lies at $y = 1$. On the basis of what we know of this function we can say that its shape is similar to the blue line in Fig. 17.3, but with a positive slope at the inflection point. Its y-intercept lies at 1.

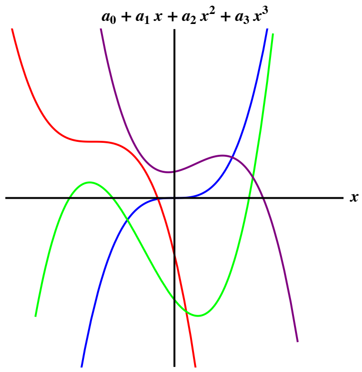


Figure 17.3: Examples of 3rd-degree polynomials.

Exercises and assignments

1. Determine the number of extrema of the following polynomials, using the derivative. Describe the general shape of each of the functions.

   a. $f(x) = x^3 - 3x^2 + 4x - 3$
   b. $f(x) = x^3 + 3x^2 - x - 4$
   c. $f(x) = x^3 - 3x^2 + 3x - 2$

17.2 *Modulus function*

The modulus, or absolute value of x is denoted by $|x|$. The modulus function is defined by:

$$\begin{aligned} &f(x) = |x| \Leftrightarrow \\ &f(x) = x \text{ if } x \geq 0 \\ &f(x) = -x \text{ if } x < 0 \end{aligned} \tag{17.17}$$

This function cannot be negative. It is symmetrical around the vertical axis, and consists of two straight line-segments. The absolute value is often used in combination with other functions. For instance, the logarithm of x only exists for $x > 0$, but the logarithm of the absolute value of x is defined for all $x \neq 0$:

$$\begin{aligned} \log(|x|) &= \log(x) \ \text{ if } x > 0 \\ &= \log(-x) \ \text{ if } x < 0 \end{aligned} \tag{17.18}$$

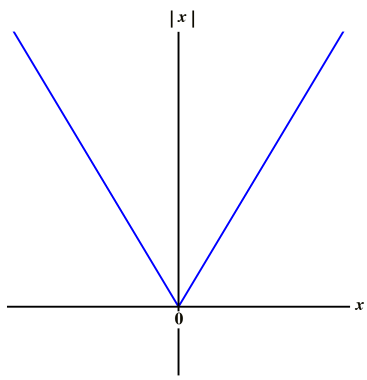


Figure 17.4: Shape of the modulus function.

Note that, since the product of positive values is positive, we have:

$$\begin{aligned} |x \cdot y| &= |x| \cdot |y| \\ |x^n| &= |x|^n \end{aligned} \tag{17.19}$$

Note that the modulus operation may be applied to other expressions than x, for example:

$$\begin{aligned} &f(x) = |x+1| \Leftrightarrow \\ &f(x) = x+1 \text{ if } x+1 \geq 0, \\ &f(x) = -(x+1) \ \text{ if } x+1 < 0 \end{aligned} \tag{17.20}$$

Exercises and assignments

1. Sketch graphs of the following functions:
   a. $f(x) = |x - 2|$
   b. $f(x) = |x + 2|$
   c. $f(x) = |x| - 2$
2. For which values of x are the following inequalities satisfied?
   a. $|x| > 1$
   b. $|x + 2| < 1$
   c. $|x - 1| \geq 2$

See chapter 18.

17.3 *Linear fractional functions*

A linear fractional function is a ratio of two linear functions:

$$f(x) = \frac{a \cdot x + b}{c \cdot x + d}, \quad \text{with } a \neq 0, c \neq 0, \text{ and } a \cdot d \neq c \cdot b \tag{17.21}$$

Note that if $ad = cb$, $a \cdot x + b$ equals a constant times $c \cdot x + d$, and the graph corresponds to a horizontal line.

The graph of this function is called a *hyperbola* . If $ad > cb$ the function increases for all x and if $ad < cb$ it is monotonically decreasing in x. There are two asymptotes: a vertical asymptote lies at the value $x = -d/c$, where the denominator in (17.21) is zero. In addition, there is a horizontal asymptote. Its value equals:

$$\lim_{x \to \infty} \frac{a \cdot x + b}{c \cdot x + d} = \frac{a}{c} \tag{17.22}$$

The expression $\lim\limits_{x \to \infty}$ means the limit, when x becomes infinitely large. Limits are considered in more detail in section 17.8. The graph of the function is point-symmetrical around the point at which the two asymptotes cross. The function has an x-intercept at $x = -b/a$ and crosses the vertical axis at $y = b/d$, provided $d \neq 0$. If $d = 0$ the vertical asymptote lies at $x = 0$, so in that case there is no y-intercept. Examples of graphs of linear fractional functions are given in Figs. 17.6 and 17.5.

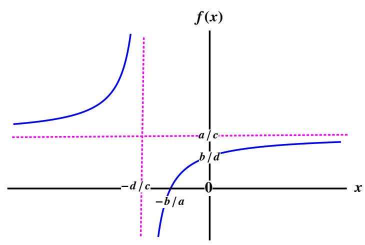


Figure 17.5: Example of the shape of a linear fractional function if $ad > cb$.

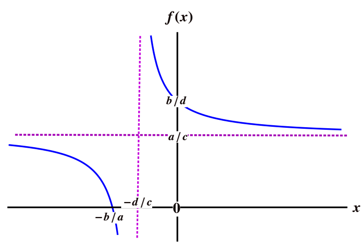


Figure 17.6: Example of the shape of a linear fractional function if $ad < cb$.

Exercises and assignments

1. For each of the following functions, determine the horizontal and vertical asymptotes, determine its intercepts, and whether it is increasing or decreasing in x, and sketch its graph.
   a. $f(x) = \dfrac{3x + 5}{2x - 4}$
   b. $f(x) = \dfrac{5x + 9}{x + 2}$
2. Determine the vertical and horizontal asymptotes of each of the following functions.
   a. $f(x) = \dfrac{\alpha \cdot x + 3}{1 - 2x}$, $\alpha \neq 0$

b. $f(x) = \dfrac{x-4}{\beta \cdot x + 3}$, $\beta \neq 0$

c. $f(x) = \dfrac{\gamma \cdot x - 5}{\beta \cdot x + 1}$, γ, $\beta \neq 0$

d. $f(x) = \dfrac{3x+2}{\alpha \cdot x - 2\beta}$, α, $\beta \neq 0$

17.4 *Power functions*

These are functions of the form:

$$f(x) = ax^b, x > 0, a \neq 0 \tag{17.23}$$

Note that when b is a whole number, this equation defines a polynomial of degree b. In that case $f(x)$ is also defined for negative x-values. For instance, when $b = 2$ the function represents a parabola.

Some examples of power functions are:

$$\begin{aligned} g(x) &= 1/x^2 && (a = 1,\ b = -2) \\ h(x) &= 3\sqrt{x} && (a = 3,\ b = 1/2) \\ r(x) &= 2 \cdot x^4 && (a = 2,\ b = 4) \end{aligned} \tag{17.24}$$

We will here consider values of b that are not necessarily whole numbers, and first examine the possible shapes of the functions for situations where $a > 0$. The effect of b is as follows:

- When $b > 1$, the function is *convex*: it increases at an accelerating rate. An example is the function $r(x)$ in (17.24).
- When $0 < b < 1$, the function is *concave*: it increases at a decelerating rate (e.g. the function $h(x)$ in (17.24)).
- When $b < 0$ the function decreases in x, and is convex (e.g. the function $g(x)$ in (17.24). The vertical and horizontal axes are both asymptotes in this case.

Note that, although the increase slows down, there is no horizontal asymptote when $b > 0$.

The different types of graphs are shown in Fig. 17.7

The following table shows the general rules for calculations with powers.

Rule	Example
$x^a \cdot x^b = x^{a+b}$	$x^2 \cdot x^3 = x^5$
$(x^a)^b = x^{a \cdot b}$	$(x^2)^3 = x^6$
$x^{-a} = 1/x^a$	$x^{-2} = 1/x^2$
$y = x^{1/a} \to y^a = x$	$y = x^{1/3} \to y^3 = x$
$1^b = 1$	$1^{0.23} = 1$
$x^0 = 1$	$0^0 = 1$
$0^a = 0$ if $a > 0$	$0^{1/2} = 0$
$0^a =$ undefined if $a < 0$	$0^{-1/3}$ is undefined

Table 17.1: Rules and definitions for powers

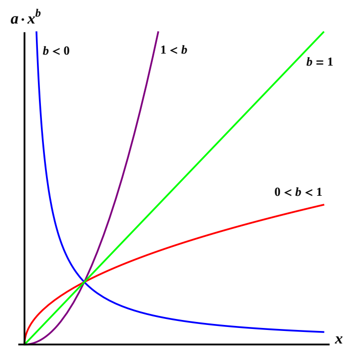


Figure 17.7: The different types of power functions. In all cases $a > 0$

The possible shapes of the functions when a is negative can be derived by taking the mirror images of the graphs in Fig. 17.7, using the x-axis as the axis of symmetry. In this case, the effects of b are:

- When $b > 1$, the function is *concave*: it decreases at an accelerating rate.

- When $0 < b < 1$, the function is *convex*: it decreases at a decelerating rate.
- When $b < 0$ the function increases in x, and is concave. The vertical and horizontal axes are both asymptotes in this case.

Exercises and assignments

1. Simplify the following expressions as far as possible:

   a. $(x^4)^2 \cdot x^5$

   b. $(x^{-3})^3$

   c. $x^5 + x^3$

   d. $x^8 \cdot x^2$

2. For each of the following functions indicate whether they are increasing or decreasing in x, and whether they are concave, or convex.

   a. $f(x) = 2 \cdot x^{-3}$

   b. $f(x) = -4 \cdot x^{1/2}$

   c. $f(x) = x^{5/4}$

   d. $f(x) = x^{-3/2}$

   e. $f(x) = -x^{7/3}$

3. For each of the following functions, indicate whether they are increasing or decreasing in x, and, when applicable give the horizontal and vertical asymptotes.

   a. $f(x) = 1 + 3 \cdot x^{-4}$

   b. $f(x) = 2 - x^2$

   c. $f(x) = 3 - \frac{1}{x^{1/3}}$

4. Simplify the following expressions as far as possible.

   (a) $(x^a)^2 \cdot x^b$

   (b) $\frac{x^2}{x^b}$

   (c) $x^{-3} \cdot x^a + x^a$

5. For which parameter values are the following functions concave? In each case we only consider non-negative values of x.

   (a) x^{a-2}

   (b) x^{a-b}

   (c) $-x^{2a}$

17.5 *Exponential functions*

An exponential function with base e has the form:

$$f(x) = e^{bx} \tag{17.25}$$

Although any positive number may be used instead of e, we will only consider this form of exponential function here.

The number e is *Euler's number*. This number cannot be represented by a finite number of decimals, but its approximate value is:

$$e \approx 2.71828 \tag{17.26}$$

The function e^x has itself as derivative, and that is why the number e occurs in many mathematical contexts. It was first used by Leonard Euler in 1736, and is named after him. The exponential function is positive for all x-values. The graph has a horizontal asymptote at the x-axis. The function is monotonically increasing in x when $b > 0$, and decreasing when $b < 0$. Its y-intercept equals $y = 1$.

The following table shows the general rules for calculations with exponentials.

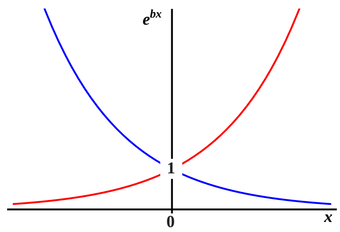


Figure 17.8: Shape of exponential functions. Blue: $b < 0$, Red: $b > 0$

Table 17.2: Rules for calculations with exponentials

Rule	**Example**
$e^a \cdot e^b = e^{a+b}$	$e^x \cdot e^{-x} = e^0 = 1$
$(e^a)^b = e^{a \cdot b}$	$\left(e^{2x}\right)^{1/x} = e^2$
$e^{-a} = 1/e^a$	$e^{-x/2} = 1/\sqrt{e^x}$
$y = e^{x/a} \Leftrightarrow y^a = e^x$	$y = e^{x/2} \Leftrightarrow y^2 = e^x$
$e^0 = 1$	

Exercises and assignments

1. Simplify the following expressions as far as possible.

   a. $e^{-3x} \cdot e^{3x}$

   b. $e^{-2x/3}$

   c. $(e^{-\frac{1}{2}x})^{-1}$

2. For each of the following functions indicate whether they are increasing or decreasing in x, give their y-intercepts, and horizontal asymptote.

   a. $f(x) = 1 + e^{2x}$

   b. $f(x) = 2 \cdot e^{-3x} - 2$

   c. $f(x) = 1 - e^{-x}$

3. For which parameter combinations are the following functions increasing in x?

   a. $f(x) = 1 + b \cdot e^{-x}$

   b. $f(x) = 3e^{a \cdot x} - 4$

   c. $f(x) = e^{a \cdot x} \cdot e^{b \cdot x}$

17.6 The natural logarithm

The natural logarithm is denoted as follows:

$$f(x) = \ln(x), x > 0 \tag{17.27}$$

In many mathematical applications, the notation $\log(x)$ is used rather than $\ln(x)$ to indicate the natural logarithm.

This function is the inverse of the exponential e^x, so:

$$\ln(x) = y \Leftrightarrow e^y = x, x > 0 \tag{17.28}$$

When y is a whole number, the natural logarithm is the number of times you have to multiply e by itself to get x. The graph of the natural logarithm is the mirror-image of the graph of e^x around the 45° line (see Fig. 17.9). It is monotonically increasing, with a vertical asymptote at the y-axis. Its intercept with the horizontal axis occurs at $x = 1$.

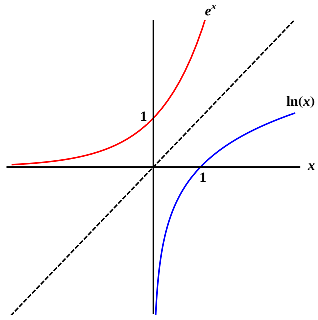


Figure 17.9: The exponential function and the natural logarithm.

The rules for calculations with natural logarithms can be derived from those for exponentials (cf. Table 17.2), for instance:

$$e^0 = 1 \Rightarrow \ln\left(e^0\right) = \ln(1) \Rightarrow \ln(1) = 0 \tag{17.29}$$

and, if we define $\alpha = \ln(a)$ and $\beta = \ln(b)$:

$$\ln(a \cdot b) = \ln\left(e^\alpha \cdot e^\beta\right) = \ln\left(e^{\alpha+\beta}\right) = \alpha + \beta = \ln(a) + \ln(b) \tag{17.30}$$

But, as you can see, this involves a lot of work. It is therefore better to practice working with the rules (for exponentials as well as logarithms) a lot, so that you can apply them readily whenever you need them. The following table lists the rules for calculations with natural logarithms:

Rule	**Example**
$\ln(a) + \ln(b) = \ln(a \cdot b)$	$\ln(2) + \ln(3) = \ln(6)$
$a \ln(x) = \ln(x^a)$	$-2\ln(x) = \ln\left(1/x^2\right)$
$\ln(1) = 0$	
$\ln(e) = 1$	

Table 17.3: Rules for calculations with natural logarithms. The rules also apply to logarithms with non-exponential base numbers, except (of course) the last rule.

Logarithms with different base numbers differ by a constant factor. As an illustration, we consider the relationship between the natural and the 10-base logarithm. The latter is defined by:

$$y = {}^{10}\log(x) \Leftrightarrow 10^y = x \tag{17.31}$$

So the natural logarithm of x can be converted into the base-10 logarithm by:

$$\begin{aligned} \ln(x) &= \ln\left(10^{{}^{10}\log(x)}\right) \\ &= {}^{10}\log(x) \cdot \ln(10) \Rightarrow \\ {}^{10}\log(x) &= \frac{\ln(x)}{\ln(10)} \end{aligned} \tag{17.32}$$

Exercises and assignments

1. Simplify the following expressions as far as possible.

   a. $\ln\left(e \cdot \sqrt{\frac{3}{2}}\right)$

   b. $2\ln(3) + \ln(4)$

   c. $\ln(\frac{1}{e})$

   d. $3\ln(e\sqrt{e})$

   e. $\ln(e \cdot 3x)$

   f. $4 + \ln(3)$

2. Express the following equations with natural logarithms:

   a. $2^4 = 16$

   b. $125 = 5^3$

   c. $9^{1/2} = 27$

   d. $81 = \left(\frac{1}{3}\right)^{-4}$

   e. $c = a^5$, with $c > 0$, $a > 0$

   f. $p^q = 3r$, with $p > 0$, $r > 0$

3. Rewrite the following expressions as single logarithms.

   a. $\ln(2) + \ln(3)$

   b. $\frac{1}{2}\ln(x) - \ln(y)$

   c. $2\ln(a) - \ln(b)$

   d. $\ln(p) + 2\ln(q)$

4. For each of the following functions determine the asymptote.

   a. $f(x) = \ln(1 - x)$

   b. $f(x) = 3\ln(x^2)$

   c. $f(x) = \ln(2 + x)$

17.7 *Trigonometric functions*

Trigonometric functions are based on the unit circle: this is a circle with a radius equal to one, centred at the coordinates $(0, 0)$ (see Fig. 17.10). If we choose a point that lies on the circle, and α is the length of the arc, then the coordinates of that point correspond to respectively $\cos(\alpha)$ and $\sin(\alpha)$ (as illustrated in the figure). The value α can also be considered as a measure of the angle of the line segment from $(0, 0)$ to the chosen point on the arc with the horizontal axis, expressed in *radians*. Thus, a radian is a unit of measurement for an angle, that relates the angle to the arc length on a circle. Since the perimeter of a unit circle is 2π, 360° corresponds to 2π radians.

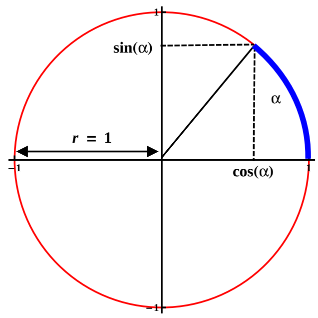


Figure 17.10: The unit circle

The value of α increases as the point on the circle moves counter clockwise. Negative values of α correspond to moving the chosen point around the circle in the other direction (clockwise).

Like the number e, π is a constant that cannot be expressed in a finite number of decimals. Its approximate value is $\pi \approx 3.14159$

If we start with the chosen point on the x-axis, at coordinates $(1,0)$, the arc length equals zero. When the point moves to the coordinates $(0,1)$, on the vertical axis, the arc length equals $\pi/2$. At the point $(-1,0)$ it is π, and at $(0,-1)$ it is $3\pi/2$. When we return to $(0,1)$ again, it equals 2π. If we continue, the arc length becomes larger than 2π. As you can see, the values of the sine and cosine functions will be repeated after each full revolution. They are so-called *periodic functions*. Both functions have a *period length* of 2π. The functions have a range of -1 to 1. The *amplitude* of the function corresponds to half the distance between these values, i.e. both the sine and the cosine have an amplitude of 1. Graphs of both functions are given in Fig. 17.11.

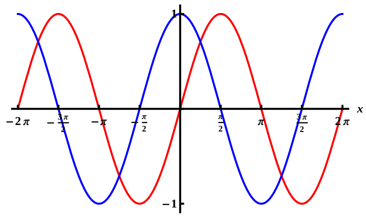


Figure 17.11: Red: $\sin(x)$, Blue: $\cos(x)$

There are several other trigonometric functions, that are all derived from the sine and the cosine. One of them is the tangent:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \tag{17.33}$$

The tangent of an angle α equals the slope of a line that has an angle of α radians with the x-axis. The tangent function has a period length of π. It has vertical asymptotes at values $x = \frac{\pi}{2} + k \cdot \pi$, where k is any whole number (see Fig. 17.12).

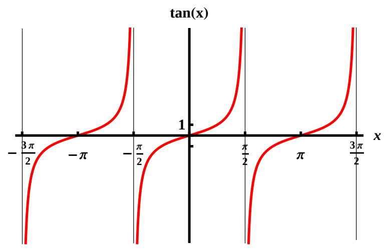


Figure 17.12: $\tan(x)$

Obviously, sine and cosine functions are related. From figure 17.11 you can see, for instance, that if you shift the cosine to the right over a distance of $\pi/2$, you get the sine function. These, and other rules that may be derived from 17.10 are summarised in the table below.

Table 17.4: Rules for sine and cosine functions

Sine	**Cosine**
$\sin(x+2\pi) = \sin(x)$	$\cos(x+2\pi) = \cos(x)$
$\sin(-x) = -\sin(x)$	$\cos(-x) = \cos(x)$
$\sin(\pi - x) = \sin(x)$	$\cos(2\pi - x) = \cos(x)$
$\sin\left(x+\frac{\pi}{2}\right) = \cos(x)$	$\cos\left(x+\frac{\pi}{2}\right) = -\sin(x)$
$\sin(a \pm b)$	$\cos(a \pm b)$
$= \sin(a)\cdot\cos(b)$	$= \cos(a)\cdot\cos(b)$
$\pm\cos(a)\cdot\sin(b)$	$\mp\sin(a)\cdot\sin(b)$
$(\sin(x))^2 + (\cos(x))^2 = 1,$ for all x	

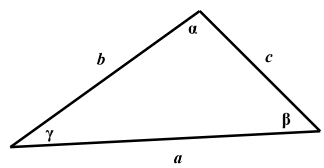


Figure 17.13: Definition of symbols used in the sine- and cosine rules.

In addition, for any triangle the following rules hold for the relationships between sines, cosines, angles, and sides (symbols as in Fig. 17.13). The *sine rule*:

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)} \tag{17.34}$$

and the *cosine rule*:

$$c^2 = a^2 + b^2 - 2a \cdot b \cdot \cos(\gamma) \tag{17.35}$$

Multiplication of a sine or cosine function with a constant affects its amplitude. For instance, the function $f(x) = 2\sin(x)$ fluctuates from -2 to 2, so it has an amplitude of 2. Multiplication of the argument inversely affects the period of the function, for instance $\sin(2x)$ has a period of π, and $\sin(x/2)$ has a period of 4π.

Exercises and assignments

1. Rewrite the following functions as expressions with $\sin(x)$ and/or $\cos(x)$.

   a. $f(x) = \sin(-x)$
   b. $f(x) = \cos(-x)$
   c. $f(x) = \sin(x + \frac{1}{2}\pi)$
   d. $f(x) = \cos(x + \frac{1}{2}\pi)$
   e. $f(x) = \sin(x + \pi)$
   f. $f(x) = \cos(x + \pi)$

2. Give the amplitudes and period lengths of the following functions:

   a. $f(x) = 3\sin(x) + 1$
   b. $f(x) = 2 - 2\cos(x)$
   c. $f(x) = \sin(5x)$
   d. $f(x) = 2\cos(x/3)$
   e. $f(x) = 3 + a \cdot \cos(x)$
   f. $f(x) = a + \sin(b \cdot x)$

3. Give the ranges of the following functions:

   a. $f(x) = 2 + \sin(x)$
   b. $f(x) = 1 - \cos(3 \cdot x)$
   c. $f(x) = 3\sin(x/4)$
   d. $f(x) = a + \sin(2x)$
   e. $f(x) = 1 + b \cdot \cos(x),\ \text{with } b > 0$
   f. $f(x) = a \cdot \cos(x/b) - 2,\ \text{with } a > 0$

17.8 Limits

Sometimes mathematical expressions are not defined for certain values of a variable or parameter. For instance, the value of the following function is undefined when $x = 2$:

$$f(x) = \frac{x^2 - x - 2}{x - 2} \tag{17.36}$$

The function does have a valid value, however, for any x other than 2, even when x lies very close to 2. The following table shows some examples:

x	$f(x)$
1.5	2.5
1.9999	2.9999
2.05	3.05
2.00001	3.00001

Table 17.5: Some values of the function in (17.36).

As demonstrated, choosing values of x close to 2 results in function values close to 3. In addition, the closer x gets to 2, the closer the function value gets to 3. The mathematical notation for this is:

$$\begin{aligned}&\lim_{x\to 2}\frac{x^2-x-2}{x-2}=3\\&\text{or, equivalently :}\\&\lim_{x\to 2}f(x)=3\end{aligned}\tag{17.37}$$

In words: 'the limit of $f(x)$ as x approaches 2 equals 3'. It can be shown quite easily that this should be the case, since the expression in the numerator of (17.37) can be factorised into $(x+1)(x-2)$, and divided by $x-2$ this leaves $(x+1)$, which indeed results in 3 at the value $x=2$.

Note however, that, although the function $g(x) = x+1$ is nearly the same as $f(x)$ they are formally different: whereas $g(x)$ does exist in the point $x=2$, $f(x)$ is not defined at this point.

In the previous example it does not matter whether we let x approach the value 2 from above, i.e. starting from a value $x>2$ and decreasing x, or from below, starting with a value of $x<2$, and increasing its value. A limit from above is denoted by an arrow pointing downwards (x decreases), and a limit from below by an arrow pointing upwards. in this example we have:

$$\lim_{x\downarrow 2}f(x)=\lim_{x\uparrow 2}f(x)=\lim_{x\to 2}f(x)=3\tag{17.38}$$

Discontinuous functions, such as in Fig. 17.14 have different limits from above and below. For the function in this figure we cannot speak of 'the limit' at $x=2$.

We can only speak of a 'limit' when limits from above and below are equal.

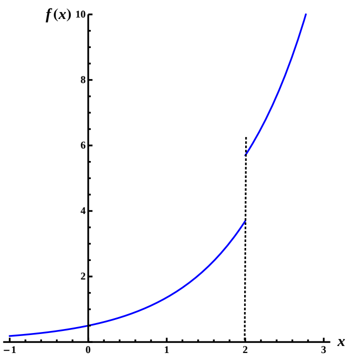


Figure 17.14: Example of a function with a discontinuity at $x=2$.

There are also functions that only have limits from above or below. For instance, $\ln(x)$ is only defined for positive values of x, and:

$$\lim_{x\downarrow 0}x\ln(x)=0\tag{17.39}$$

This is shown below in Eq. (17.40)

The previous examples concern limits when x approaches a fixed value. Another type of limits are those where x becomes infinitely large or infinitely negative. Such limits correspond to horizontal asymptotes in graphs (see e.g. Fig. 17.5).

There are different ways of finding the values of limits. What works the best depends on the type of limit and the type of function that we are considering. Here, we will consider a few (but not all) possible methods, that are illustrated by examples. In addition, Table 17.6 gives an overview of the most common limits.

Numerical approximation

A non-elegant, but sometimes useful way to find limits is simply computing values by means of a calculator or computer. For limits

The value of a limit may officially not be infinite: if $f(x)$ becomes infinite the limit does not exist. For ease of notation, however, we will sometimes write, for example: $\lim_{x\to 0}1/x=\infty$.

as $x \to \infty$ just fill in an enormously large number, such as 10^{10}. For limits when $x \uparrow 0$, take $-1/100,000$ etc. This method cannot always be applied. For instance, when there are unknown parameters it does not work.

Rearranging an expression
Limits for x approaching a constant value may sometimes be found by writing an expression in a different way. This was already illustrated in a previous example in (17.36). Another example is:

$$\lim_{x \downarrow 0} x \cdot \ln(x) = \lim_{x \downarrow 0} \ln(x^x) = \ln\left(0^0\right) = \ln(1) = 0 \tag{17.40}$$

Dividing by the highest power of x
To derive limits of fractions as x approached (plus or minus) infinity, divide numerator and denominator by the highest power of x that occurs in the expression. Then let x go to (plus or minus) infinity. Expressions that previously contained a lower power of x will go to zero. Here are some examples:

$$\lim_{x \to \infty} \frac{x^2 + 10}{100x + 0.1x^2} = \lim_{x \to \infty} \frac{1 + \frac{10}{x^2}}{\frac{100}{x} + 0.1} = \frac{1}{0.1} = 10 \tag{17.41}$$

$$\lim_{x \to \infty} \frac{x^2 + a \cdot x^3}{b \cdot x + x^3} = \lim_{x \to \infty} \frac{\frac{x^2}{x^3} + a}{\frac{bx}{x^3} + 1} = \lim_{x \to \infty} \frac{\frac{1}{x} + a}{\frac{b}{x^2} + 1} = a \tag{17.42}$$

and:

$$\lim_{x \to \infty} \frac{a + x}{bx^2} = \frac{\frac{a}{x^2} + \frac{1}{x}}{b} = 0 \tag{17.43}$$

If the highest power of x occurs only in the numerator, the limit does not exist, for example:

$$\lim_{x \to \infty} \frac{ax^2 + c}{bx} = \infty \tag{17.44}$$

Relative rates of increase

There are some general rules of thumb for expressions containing $\ln(x)$, x^a, e^{bx}, that may be useful for finding limits. The rates of increase of these functions can be ordered as follows:

The exponential function, e^{bx}, $(b > 0)$, increases the fastest.

The power function, x^a, $(a > 0)$, has the next fastest rate of increase.

The logarithmic function, $\ln(x)$ has the lowest rate of increase.

Here are some examples. In all cases $a > 0$ and $b > 0$:

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} = 0$$
$$\lim_{x \to \infty} \frac{e^x}{x^a} = \infty$$
$$\lim_{x \to \infty} x^a e^{-bx} = 0 \tag{17.45}$$

for $a < 0$ $\lim\limits_{x\to\infty} x^a = 0$	for $a < 0$ $\lim\limits_{x\to\infty} e^{ax} = 0$	for all $k \in \mathbb{R}$ and $a < 0$ $\lim\limits_{x\to\infty} x^k e^{ax} = 0$
for all $n \in \mathbb{N}$ $\lim\limits_{n\to\infty} a^n = 0$ if $-1 < a < 1$	for all $k \in \mathbb{R}$, and $a < 0$ $\lim\limits_{x\to\infty} x^a = 0$	for all $k \in \mathbb{R}$, and $a > 0$ $\lim\limits_{x\downarrow 0} x^a (\ln(x))^k = 0$
$\lim\limits_{x\to 0} \frac{\sin(x)}{x} = 1$	$\lim\limits_{x\to 0} \frac{\tan(x)}{x} = 1$	$\lim\limits_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$
$\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$	$\lim\limits_{x\to 0} \frac{e^x - 1}{x} = 1$	for $a > 0$ $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = \ln(a)$
$\lim\limits_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$	$\lim\limits_{x\to\infty} x^{\frac{1}{x}} = 1$	for all $a \in \mathbb{R}$ $\lim\limits_{x\to\infty} \left(\frac{a}{x}\right)^x = 0$

Table 17.6: An overview of the most common limits.

Exercises and assignments

1. Find the following limits.

   a. $\lim\limits_{x\to\infty} \dfrac{x^2 - 1}{2x - 3x^2}$

   b. $\lim\limits_{n\to\infty} \left(\dfrac{1}{3}\right)^n$

18
Solving equations and inequalities

Equations and inequalities indicate the relationship between two mathematical quantities. For instance, we may state that $3 + 5 = 8$, or $3 < 4 + 1$. If one of the numbers involved is unknown, it may be indicated by a symbol, for instance $x - 6 = 7$. In that case, the value of the unknown (or 'variable') x can be inferred by solving the equation (in this case yielding $x = 13$).

The use of parameters in models is explained in section 9.2.

In many practical applications, equations contain free parameters, which are indicated by a symbol. In that case, first it should be determined which symbol represents the unknown, and then the equation is solved by expressing the unknown in terms of the other symbols. For instance, the following equality contains five symbols:

$$a \cdot x + b \cdot x = c \cdot a + d \cdot x \tag{18.1}$$

If x is the unknown, x must be expressed in terms of a, b, c, and d. If, however, a is the unknown, the equation is solved by expressing a in terms of x, b, c, d. There are many possible forms of equations and inequalities, and it is impossible to consider them all. Furthermore, not all equations can be solved analytically. In this chapter we will limit ourselves to a few examples of the type of equations that can be solved and are encountered frequently in applications.

Unless stated otherwise, we will indicate the unknown by x, and let other symbols represent parameters.

18.1 Equations: introduction

We will first consider equations, and later turn to solving inequalities. When solving an equation, the goal is to end up with an equation that has the unknown on one side of the '=' sign, and an expression containing the parameters on the other side. To acquire that, all kinds of mathematical manipulation may be performed, as long as it happens on both sides of the equality sign. For some operations, however, some care is needed. The following table lists several types of operations, and the conditions under which they are allowed. The symbol α stands for an arbitrary constant, variable, or expression.

Operation	Condition
adding α	
subtracting α	
multiplying with α	$\alpha \neq 0$
dividing by α	$\alpha \neq 0$
taking roots (squared, cubed or other)	expressions must be positive
taking logarithms	expressions must be positive
converting to exponents	

Table 18.1: Operations for solving equations

18.2 *Linear equations*

Here are some examples of linear, or *first-order equations*:

$$\begin{aligned} 2x - 3 &= 0 \\ 3x + 1 &= 4 \\ x + 1 &= 2x - 5 \end{aligned} \tag{18.2}$$

The feature that these equations have in common, and what makes them linear equations, is the fact that all components that contain the unknown variable x are linear. Some example of nonlinear equations are:

$$\begin{aligned} 3x^2 - 3 &= 1 \\ \ln(x) - 4 &= 0 \\ 3\sqrt{x} - 2 &= 1 \end{aligned} \tag{18.3}$$

The expressions in (18.2) can (after some rearrangements) all be written in the following form:

$$a \cdot x + b = 0, \text{ with } a \neq 0 \tag{18.4}$$

A linear equation has a single solution, that may be found by subtracting b on both sides and then dividing by a:

Linear equations have a single solution, and may be solved by using addition/subtraction, and multiplication/division.

$$\begin{aligned} a \cdot x + b &= 0 \\ a \cdot x &= -b && \text{(subtract } b \text{ on both sides)} \\ x &= -\frac{b}{a} && \text{(divide both sides by } a\text{)} \end{aligned} \tag{18.5}$$

After you have solved the equation, you can check your result by substituting the solution. If it is correct, the resulting equation should be correct. In this case, for instance:

$$\begin{aligned} a \cdot \left(-\frac{b}{a}\right) + b &= 0 \\ -b + b &= 0 \end{aligned} \tag{18.6}$$

Since the resulting statement is valid, the solution is indeed correct.

Sometimes a few more steps may be needed. For instance, in equation (18.1), the expressions containing x must be joined first.

This equation is solved as follows:

$$\begin{aligned}
a \cdot x + b \cdot x &= c \cdot a + d \cdot x \\
a \cdot x + b \cdot x - d \cdot x &= c \cdot a && \text{(subtract } d \cdot x \text{ on both sides)} \\
(a + b - d) \cdot x &= c \cdot a && \text{(factorise the left - hand side)} \\
x &= \frac{c \cdot a}{a + b - d} && \text{(divide both sides by } a + b - d\text{)} \\
condition &: a + b - d \neq 0 && (18.7)
\end{aligned}$$

Take care to state conditions on parameters explicitly in the solution.

Note that a condition needs to be added since it is not allowed to divide by zero. When free parameters are involved, you must always take care to state such conditions explicitly. If $a + b - d$ is zero, this equation has no solution (unless $c \cdot a$ also equals zero, in which case any value of x would satisfy the equation).

Exercises and assignments

1. Solve the following equations for x, and write down all the steps with conditions where needed.

   a. $(1 - a) \cdot x + 2x = 5$

   b. $\alpha \cdot x - 3 = 2x + 4$

   c. $-\alpha x + 3 = \gamma x - 5$

2. Solve the following equations for x, and write down all the steps with conditions where needed.

   a. $a \cdot x - x + b = c$

   b. $c \cdot x - d = a \cdot x - b$

3. Solve the following equations for ρ, and write down all the steps with conditions where needed.

   a. $a \cdot \rho - 2b = 5\rho$

   b. $x \cdot \rho - 3 = 2x + 1$

18.3 Quadratic equations

This type of equation is also called a second-order equation. Denoting the unknown variable by x, the general form of such an equation is:

$$a \cdot x^2 + b \cdot x + c = 0, a \neq 0 \tag{18.8}$$

As mentioned before, when working with parameters it is important to define the unknown variable in the equation. If a rather than x is the unknown variable, for instance, Eq. (18.8) would represent a *linear* rather than a quadratic equation, and the methods of section 18.2 should be used to solve it.

Quadratic equations may have one, two, or no solutions for $x \in \mathbb{R}$.

If we include the possibility of complex numbers, they always have solutions, but here we will limit ourselves to solutions that are real numbers.

Sometimes solutions can be found directly, through factorisation, for example:

$$\begin{aligned} x^2 - 10x + 21 &= 0 \\ (x-7)\cdot(x-3) &= 0 \\ x = 7 \text{ or } x &= 3 \end{aligned} \tag{18.9}$$

Factorisation may also be possible when there are free parameters, for example:

$$\begin{aligned} x^2 + 2\beta \cdot x + \beta^2 &= 0 \\ (x+\beta)^2 &= 0 \\ x &= -\beta \end{aligned} \tag{18.10}$$

It is, however, not always easy to find a factorisation. In that case, the *abc-formula* offers a solution. This method always works. This formula states that the solutions of equation (18.8) are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{18.11}$$

The expression under the square-root sign is called the *discriminant* of the equation. There are two different, real-valued solutions if this value is positive. If it is zero, the equation has one, real-valued solution, and if it is negative, there are no $x \in \mathbb{R}$ that satisfy the equation.

Here is an example of application of the abc-formula to an equation with a parameter:

$$\begin{aligned} &2x^2 - 4px + 1 = 0 \\ &x = \tfrac{4p \pm \sqrt{16p^2 - 8}}{4} \\ &x = p \pm \sqrt{4p^2 - 2} \end{aligned} \tag{18.12}$$

Since the parameter p turns up in the discriminant, we need to find out for which values of p there are solutions in $\mathbb{R}$. This involves solving a quadratic inequality. We will demonstrate it here. The subject of solving quadratic inequalities is treated in more detail in section 18.8.

$$\begin{aligned} &4p^2 - 2 > 0 \\ &4p^2 > 2 \\ &p^2 > \tfrac{1}{2} \\ &p > \sqrt{\tfrac{1}{2}} \text{ or } p < -\sqrt{\tfrac{1}{2}} \end{aligned} \tag{18.13}$$

Thus, the full solution of the quadratic equation in (18.12) is:

If $p > \sqrt{\frac{1}{2}}$ or $p < -\sqrt{\frac{1}{2}}$ there are two roots, corresponding to the outcomes given in (18.12).

If $p = \pm\sqrt{\frac{1}{2}}$ the discriminant in (18.12) is zero, so there is one solution: $x = p$.

If $-\sqrt{\frac{1}{2}} < p < \sqrt{\frac{1}{2}}$ there are no real-valued solutions for x.

The equation does have complex-valued solutions. In this chapter, however, we only consider solutions $x \in \mathbb{R}$.

Exercises and assignments

1. Solve the following equations for x:

   a. $2x^2 - 3x = 1$

   b. $x^2 = x + 2$

   c. $3x^2 + 2x = x + 2$

2. Solve the following equation for x. Add conditions on the parameters where necessary: $ax^2 + x = 2b$

3. Solve the following equation for x, and list all possible outcomes for different values of p:
   $p \cdot x^2 + 2x - 1 = 0, p \neq 0$

4. Solve the following equations for α.

   a. $2\alpha^2 - 4\alpha + 2 = 0$

   b. $x^2 \cdot \alpha - 2x + 3 = 0$

18.4 Fractional equations

These are equations where the unknown variable occurs in the denominator of fractions, for instance:

$$\frac{1}{x-a} = 2 + b,\ x \neq a \tag{18.14}$$

The first step to solving such equations is to get x out of the denominator(s). This is done by multiplying with the expression containing x:

Note that there must be conditions on the values of x in this case, since the denominator cannot be equal to zero. Such conditions must be stated explicitly before solving the equation, otherwise you may end up with invalid solutions.

$$\begin{aligned} \frac{1}{x-a} &= 2 + b,\ x \neq a \\ 1 &= (2+b) \cdot (x-a) \quad \text{(multiply both sides by } (x-a)) \end{aligned} \tag{18.15}$$

In this case we end up with a linear equation that can be solved using the methods in section 18.2. If there are several fractions with expressions containing x in the denominator, repeated multiplications are carried out, for example:

$$\begin{aligned} \frac{1}{x-b} &= \frac{1}{x} + 2,\ x \neq 0 \text{ and } x \neq b \\ 1 &= \frac{x-b}{x} + 2 \cdot (x-b) && \text{(multiply with } (x-b) \text{ on both sides)} \\ x &= x - b + 2 \cdot (x-b) \cdot x && \text{(multiply with } x \text{ on both sides)} \end{aligned} \tag{18.16}$$

In this case the resulting equation is quadratic, and needs to be solved with the methods in section 18.3

Exercises and assignments

1. Solve the following equations for x. Write down all steps and add conditions where necessary.

   a. $\frac{1}{x} - 3 \cdot x = 0$

   b. $\frac{1}{x+2} - 3 = 0$

   c. $\frac{1}{x-1} = -3x$

   d. $\frac{x+2}{x-1} = 3$

2. Solve the following equations for x. Write down all steps and add conditions where necessary.

   a. $\frac{1}{x+a} = -3$

   b. $\frac{1}{a \cdot x} = \frac{b}{x} + c$

   c. $\frac{x}{x+a} - 3x = 0$

3. Solve the following equations for x. Write down the steps and add conditions where necessary.

   a. $\frac{1}{x+a} - 2x = 0$

   b. $\frac{1}{x-a} + \frac{b}{x} = 0$

18.5 Equations involving a root

When equations involve expression with x under a root sign, care must be taken, since such expressions must be positive. It is not always possible to solve such equations. Here we will only consider an example that can be solved, to illustrate the approach:

$$3x - 2\sqrt{1-x} = 1, \; x \leq 1 \tag{18.17}$$

Since the expression under the root sign must be positive, the possible solutions are constrained. It is important to write up such conditions before starting to solve the equation. The next step is to rearrange the equation in such a way that the root occurs on one side, and the rest on the other side of the equality sign:

Note the condition on x!

$$\begin{aligned} 3x - 1 - 2\sqrt{1-x} &= 0 && \text{(subtract 1 on both sides)} \\ 3x - 1 &= 2\sqrt{1-x} && \text{(add } 2\sqrt{1-x} \text{ on both sides)} \end{aligned} \tag{18.18}$$

Now we must impose another condition on x: since the outcome of a root is always positive, the left-hand side must also be positive, so:

$$\begin{aligned} 3x - 1 &= 2\sqrt{1-x} && \text{condition: } 3x - 1 \geq 0, \text{ so } x \geq 1/3 \\ (3x-1)^2 &= 4\,(1-x) && \text{(raise both sides to the power 2)} \end{aligned} \tag{18.19}$$

Note the extra condition on x!

Working out the multiplication gets rid of the brackets, and we end up with a quadratic equation, that may be solved with the methods in section 18.3. The final solutions need to be tested, since they have to meet the two conditions stated in (18.17) and (18.19). Combining these gives:

$$1/3 \leq x \leq 1 \tag{18.20}$$

It is left as an exercise for the reader to show that the solutions of the quadratic equation in (18.19) are equal to -0.477 and 0.699 (rounded off to three decimals). Since only the second value satisfies the requirement, there is only one solution to the original equation:

$$x \approx 0.699 \tag{18.21}$$

The exact solution is $x = \frac{1+2\sqrt{7}}{9}$.

As illustrated by this example, the solution method for an equation containing a root starts by rearranging the expression, in order to get an equation where the root is on one side and the other terms are on the other side of the equality sign. After that has been accomplished, both sides are raised to the power two.

Note that this solution method may not always be possible. For instance, it may not be possible if an equation contains multiple roots of different expressions containing the unknown.

When unspecified parameters are involved, conditions may need to be imposed on them too. Here is an example:

$$\sqrt{a \cdot x + 2} - b = 0 \tag{18.22}$$

Since the expression under the root sign needs to be nonnegative, the first condition is that:

$$a \cdot x \geq -2 \tag{18.23}$$

Adding b to both sides gives:

$$\sqrt{a \cdot x + 2} = b \tag{18.24}$$

So, since the root must be nonnegative, another condition is that $b \geq 0$. Proceeding with the solution we find:

$$\begin{aligned} a \cdot x + 2 &= b^2 && \text{raise both sides to power 2} \\ a \cdot x &= b^2 - 2 && \text{subtract 2 on both sides} \\ x &= \frac{b^2 - 2}{a} && \text{divide both sides by } a \\ & && \text{condition: } a \neq 0 \end{aligned}$$

Since the outcome needs to satisfy the condition in (18.23):

$$a \cdot \frac{b^2 - 2}{a} \geq -2 \Rightarrow b^2 - 2 \geq -2 \tag{18.25}$$

So $b^2 \geq 0$. Since this is always true it poses no extra conditions. The outcome, including all conditions on the parameter values is:

$$x = \frac{b^2 - 2}{a}, \text{ with } a \neq 0 \text{ and } b \geq 0 \tag{18.26}$$

Exercises and assignments

1. Solve the following equations. Write down all the steps, and the conditions where needed.

   a. $\sqrt{x-2} = 3$

   b. $\sqrt{x+1} - 2\sqrt{x-1} = 0$

2. Solve the following equations. Write down the steps and add conditions where necessary.

   a. $\sqrt{x} = 2 - x$

   b. $\sqrt{x} - 3 = 3\sqrt{x} - 2$

3. Solve the following equations for x. Write down the steps and add conditions where necessary.

   a. $\sqrt{a \cdot x - 1} - 3 = 0$

   b. $\sqrt{x - a} = 3b$

18.6 *Equations involving exponentials and logarithms*

Many modelling applications involve equations containing exponential functions or (natural) logarithms. Such equations cannot always be solved analytically. We will here show a few examples that can be solved, to illustrate solution methods.

If an equation involves one exponential function, and the unknown x occurs in the exponent and nowhere else, it can be solved. Here is an example:

$$e^{ax+b} = c, c > 0 \tag{18.27}$$

Note the condition on c. If $c \leq 0$ there is no solution

To solve this equation, take logarithms on both sides:

$$ax + b = \ln(c) \tag{18.28}$$

Note that the result is a linear equation: since c is a constant, $\ln(c)$ is a constant as well. Thus, the equation can be solved by using the methods in section 18.2. The result is (derive this as an exercise):

$$x = \frac{\ln(c) - b}{a}, a \neq 0 \tag{18.29}$$

Equations that only contain two exponential functions may also be used by taking logarithms, for instance:

$$e^{2x} = a \cdot e^{3x-1}, a > 0 \tag{18.30}$$

Taking natural logarithms on both sides gives:

$$2x = \ln(a) + 3x - 1, \tag{18.31}$$

which is, again, a linear equation. The solution is left as an exercise to the reader.

Solving equations with logarithms proceeds in a similar way, only here expressions on both sides of the equation are transformed by taking exponents, for instance:

$$\begin{aligned}
\ln(3\cdot x) &= b, && \text{condition: } x > 0 \\
e^{\ln(3\cdot x)} &= e^b && \text{take exponents on both sides} \\
3\cdot x &= e^b \\
x &= \frac{e^b}{3} && (18.32)
\end{aligned}$$

Note that the outcome is positive regardless the value of b, so the condition on x is satisfied

Exercises and assignments

1. Derive the result in (18.29)
2. Solve (18.31) for x.
3. Solve for x: $ae^{bx} = \frac{1}{2}, a > 0$

18.7 *Equations involving trigonometric functions*

Here we will consider equations of the form:

$$f(x) = c, \tag{18.33}$$

where $f(x)$ is one of the trigonometric functions considered in section 17.7 and c a constant. Since trigonometric functions are periodic, such equations may have infinitely many solutions. As an example, consider:

$$\sin(x) = 1/2 \tag{18.34}$$

One of the solutions is $x = \frac{\pi}{6}$, but since $\sin(x + 2\pi) = \sin(x)$, any value $x = \frac{\pi}{6} + k\cdot 2\pi$, where k is a whole number is also a solution. Furthermore, since $\sin(\pi - x) = \sin(x)$, $x = \frac{5\pi}{6}$ is also a solution. The complete set of solutions is:

$$x = \frac{\pi}{6} + k\cdot 2\pi, \text{ or } x = \frac{5\pi}{6} + k\cdot 2\pi,\ k \in \mathbb{Z} \tag{18.35}$$

Similar considerations apply to solutions for of other trigonometric functions, for instance, the solutions of:

$$\cos(x) = 1/2 \tag{18.36}$$

are

$$x = \frac{\pi}{3} + k\cdot 2\pi, \text{ or } x = \frac{-\pi}{3} + k\cdot 2\pi,\ k \in \mathbb{Z} \tag{18.37}$$

The tangent function has a period of π rather than 2π, so we get for instance:

$$\tan(x) = 1 \Rightarrow x = \frac{\pi}{4} + k\cdot\pi,\ k \in \mathbb{Z} \tag{18.38}$$

It should furthermore be noted that sine and cosine functions have a limited range, for instance, the equation:

$$\sin(x) = a \tag{18.39}$$

has no solution if a is less than -1 or greater than 1.

Exercises and assignments

1. Solve the following equations for x:

   a. $\cos(2x) = 1/2$

   b. $\sin(x + 3) = -1$

   c. $\tan(x/3) = 1$

   d. $2 \cdot \cos(x) = 1$

2. Solve the following equations for x:

   a. $\cos(a \cdot x) = 1, a \neq 0$

   b. $\sin(a \cdot x + b) = -1, a \neq 0$

18.8 Inequalities

Inequalities are statements concerning an unknown (as before, denoted by x) involving an inequality sign rather than an '=' sign. Solutions of equations are usually just one or a few x-values. The solutions of inequalities are usually intervals. In some cases these may be infinitely large. For instance, the solution of the inequality $x - 1 < 0$ consists of all values smaller than 1.

Inequality signs are $>, \geq, <, \leq$.

Solving an inequality is very similar to solving an equation, using operations listed in Table 18.1. The same conditions apply, for instance division by zero, or taking the logarithm of negative numbers is not allowed. An extra complication is that some operations, such as division by or multiplication with a negative number affect the inequality sign. To illustrate this, Fig. 18.1 shows what happens if you multiply all numbers on a number line by -2. The result is, for instance that, whereas 3 is larger than 1, -6 is smaller than -2, so $-2 \cdot 3 < -2 \cdot 1$. In this section we will illustrate how to solve inequalities by means of a few, typical, examples. The possible operations, and their consequences for inequalities are listed in Table 18.2, where the symbol α stands for an arbitrary constant, variable, or expression.

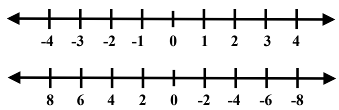


Figure 18.1: Top: number line. Bottom: all numbers multiplied by -2.

Table 18.2: Operations for solving inequalities

Operation	Condition and consequences
adding α	
subtracting α	
multiplying with α	$\alpha \neq 0$. If $\alpha < 0$ the inequality sign reverses
dividing by α	$\alpha \neq 0$. If $\alpha < 0$ the inequality sign reverses
taking roots (squared, cubed or other)	expressions must be positive
taking logarithms	expressions must be positive
converting to exponents	

18.8.1 Linear inequalities

An example of a linear inequality is:

$$4 - 3x > x + 1 \tag{18.40}$$

Solving the inequality is very similar to solving an equation:

$$\begin{aligned}
4 - 3x &> x + 1 \\
-3x &> x - 3 && \text{(subtract 4 from both sides)} \\
-4x &> -3 && \text{(subtract } x \text{ from both sides)} \\
4x &< 3 && \text{(multiply both sides by } -1) \\
x &< \frac{3}{4} && \text{(divide both sides by 4)} \qquad (18.41)
\end{aligned}$$

Note that in the third step the inequality sign changes, since both sides are divided by a negative number.

When an inequality contains free parameters, different possibilities may need to be explored, for instance, here is the same inequality, with 3 replaced by an unspecified parameter a:

$$4 - a \cdot x > x + 1 \tag{18.42}$$

The first steps of the solution are similar to those in (18.41):

$$\begin{aligned}
4 - a \cdot x &> x + 1 \\
-a \cdot x &> x - 3 && \text{(subtract 4 from both sides)} \\
-a \cdot x - x &> -3 && \text{(subtract } x \text{ from both sides)} \\
a \cdot x + x &< 3 && \text{(multiply both sides by } -1) \\
x\,(a + 1) &< 3 && \qquad (18.43)
\end{aligned}$$

Before we proceed and divide by $a + 1$, however, we must distinguish two different situations: if $a + 1$ is positive, the inequality sign stays the same, if it is negative, it changes. Finally, if $a = -1$ the left-hand side is zero, regardless of x. Thus, the complete solution is:

$$\begin{aligned}
x &< \frac{3}{a+1} && \text{if } a > -1 \\
x &> \frac{3}{a+1} && \text{if } a < -1 \\
x &\in \mathbb{R} && \text{if } a = 1 \qquad (18.44)
\end{aligned}$$

An example of a linear inequality that contains several parameters is:

$$a \cdot x + b \leq c \cdot x + d \tag{18.45}$$

As with linear equations, the first step is to rearrange terms and acquire an expression with the unknown, x, on one side of the inequality sign, and the rest on the other side:

$$\begin{aligned}
a \cdot x + b &\leq c \cdot x + d \\
a \cdot x + b - c \cdot x &\leq d && \text{(subtract } c \cdot x \text{ on both sides)} \\
a \cdot x - c \cdot x &\leq d - b && \text{(subtract } b \text{ on both sides)} \\
x \cdot (a - c) &\leq d - b && \text{(factorise left - hand side)} \qquad (18.46)
\end{aligned}$$

Up to now, the operations that were carried out did not affect the inequality sign. The next step, however, involves a division by $a - c$. Here we need to distinguish different situations:

$$\begin{aligned} &\text{If } a - c < 0: && x \geq \frac{d-b}{a-c} \\ &\text{If } a - c = 0: && \text{no solution if } d - b < 0,\ x \in \mathbb{R} \text{ if } d - b \geq 0 \\ &\text{If } a - c > 0: && x \leq \frac{d-b}{a-c} \end{aligned} \tag{18.47}$$

Exercises and assignments

1. Solve the following inequalities for x. Find all solutions:
   a. $x + b > 2x - 3$
   b. $c \cdot x + 1 \leq 2x + 3$
   c. $d \cdot x - 2 \geq 3x + 1$
2. Solve the following inequalities for x. Find all solutions:
   a. $a \cdot x + 1 > x - b$
   b. $a \cdot x + 2 \leq -c \cdot x + 3$

18.8.2 Quadratic inequalities

Quadratic inequalities involve quadratic powers of x, for example:

$$x^2 - 3x < 2 \cdot (3 - x) \tag{18.48}$$

Subtracting $2\,(3 - x)$ on both sides, and carrying out some further manipulations gives:

$$\begin{aligned} x^2 - 3x - 2 \cdot (3 - x) &< 0 \\ x^2 - 3x - 6 + 2x &< 0 \\ x^2 - x - 6 &< 0 \end{aligned} \tag{18.49}$$

By factorisation, or using the abc-formula (see 18.3, it can be derived that the expression on the left-hand side is zero when:

$$x = 3 \text{ or } \mathrm{x} = -2 \tag{18.50}$$

Since, furthermore the expression on the left-hand side in equation (18.49) specifies a u-shaped parabola (see section 17.1.2), it is negative for values of x that lie in-between the intercepts, so the solution of the inequality is:

$$-2 < x < 3 \tag{18.51}$$

When free parameters are involved, there are a few additional steps in solving equations. As an illustration, let's consider the inequality:

$$x^2 - 3x < h \cdot (3 - x) \tag{18.52}$$

Note that the inequality in (18.48) is a special case of this inequality, with $h = 2$. Rearranging the inequality in a similar way as before gives:

$$\begin{aligned} x^2 - 3x - h \cdot (3 - x) &< 0 \\ x^2 - 3x - 3h + h \cdot x &< 0 \\ x^2 + (h - 3) \cdot x - 3h &< 0 \end{aligned} \tag{18.53}$$

By factorisation, or using the abc-formula (see 18.3, it can be derived that the expression on the left-hand side is zero when:

$$x = 3 \text{ or } x = -h \tag{18.54}$$

As before, the expression on the left-hand side in equation (18.53) specifies a u-shaped parabola (see section 17.1.2), and therefore it is negative for values of x that lie in-between the intercepts. Now, however, it depends on the value of h whether such intercepts exist, and which of the two is the smallest. The situation considered in (18.48) corresponds to a situation where $h = 2$, and since $-2 < 3$, the solution was $-2 < x < 3$. If $h = -4$, however, the two intercepts are 3 and 4, and the solution would be $3 < x < 4$. If $h = -3$, the expression in (18.53) is never negative. The fully specified solution is:

If $h = -3$ the expression is zero when $x = 3$, and larger than zero otherwise.

$$\begin{aligned} \text{If } -h < 3 \ (\text{so } h > -3): &\quad \text{solution } -h < x < 3 \\ \text{If } -h > 3 \ (\text{so } h < -3): &\quad \text{solution } 3 < x < -h \\ \text{If } -h = 3: &\quad \text{no solution exists} \end{aligned} \tag{18.55}$$

Exercises and assignments

1. Solve the following inequalities for x. Examine solutions for all possible parameter values:

   a. $a \cdot x^2 > 2$
   b. $a \cdot x^2 + 3 > 2$

2. Solve the following inequality for x. Examine solutions for all possible parameter values:
   $a \cdot x^2 + 5x - 2 > 0, a \neq 0$

18.9 Systems of equations

Sometimes there are more than one unknown variable. To find the values of these unknowns, a system of equations needs to be solved. In general, the number of equations in such a system is equal to the number of unknowns.

If there are less equations than unknowns, there are no unique solutions. In such cases we may find combinations of the unknowns that satisfy the equations. If there are more equations than unknowns, either one or more of the equations is superfluous, or it is not possible to find a solution.

The way to solve a system of equations is usually to eliminate unknowns until you are left with a single equation, with one unknown, which can be solved with methods explained in previous sections. The solution can then be substituted in the other equations, after which the following unknown can be found, etc. Examples are given in the next section.

18.9.1 Linear systems of equations

In a linear system all the equations contain linear combinations of the unknown variables. An example of a linear system with two unknowns, x and y is:

$$\begin{aligned} 3x + 2y &= 4 \qquad (1) \\ x - y &= 5 \qquad (2) \end{aligned} \tag{18.56}$$

The equations are numbered for ease of reference. Using equation (2) to express y in x gives:

$$x - 5 = y \tag{18.57}$$

Substituting this result in (1) and solving for x gives:

$$\begin{aligned} 3x + 2(x - 5) &= 4 \\ 5x - 10 &= 4 \\ 5x &= 14 \\ x &= \frac{14}{5} \end{aligned} \tag{18.58}$$

Substituting this result in (18.57) provides the value of y:

$$y = \frac{14}{5} - 5 = -\frac{11}{5} \tag{18.59}$$

When there are more than one unknown, the procedure is similar. In that case, choose one of the variables to be eliminated first, express that variable in the two others. Substitution will give a system of two equations with two unknowns, which can be solved as demonstrated above. The solution is then used to find the value of the third, initially eliminated unknown. Here is an example:

$$\begin{aligned} x + y + z &= 0 \qquad (1) \\ 2x - y &= 6 \qquad (2) \\ x + z &= -4 \qquad (3) \end{aligned} \tag{18.60}$$

Let's first eliminate y, by using equation (2), and substituting the result in equation (1):

$$\begin{aligned} y &= 2x - 6 \\ x + 2x - 6 + z &= 0 \\ 3x - 6 + z &= 0 \end{aligned} \tag{18.61}$$

We now have a system of two equations left:

$$\begin{aligned} 3x - 6 + z &= 0 \\ x + z &= -4 \end{aligned} \tag{18.62}$$

The solution of these equations is $x = 5,\ z = -9$ (check this yourself).

Solving linear systems of equations is straightforward, but may become quite tedious as the number of unknowns increases. Usually, computer programs, and matrix algebra (see chapter 21) are used to find solutions.

Note that there is not always a solution. A (simple) example is the following system:

$$\begin{aligned} x + y &= 2 \\ x + y &= 3 \end{aligned} \tag{18.63}$$

There are also systems with an infinite number of solutions, for example:

$$\begin{aligned} x + y &= 2 \\ 2x + 2y &= 4 \end{aligned} \tag{18.64}$$

A twodimensional linear system of equations may have one, zero, or an infinite number of solutions. It is not possible for such a system to have, say, 2 solutions. For instance there is no system of two linear equations that has the solution $x = 1$ and $y = 2$ as well as the solution $x = 0$ and $y = 3$. Nonlinear systems of equations, however, may have such solution sets (some examples are given in section 18.9.2.

This may be illustrated graphically: two lines either have one intersection point, or none (when they are parallel to each other), or they are the identical.

Exercises and assignments

1. Solve the following systems of equations for x and y:

   a.
   $$\begin{aligned} 3x &= 2y + x + 1 \\ x + y &= 4 \end{aligned}$$

   b.
   $$\begin{aligned} x - y &= -1 \\ 6x - 7y &= 3 \end{aligned}$$

   c.
   $$\begin{aligned} x - y &= x + 3y \\ -2y &= 7 + x \end{aligned}$$

2. Solve the following systems of equations for x and y:

   a.
   $$\begin{aligned} x - y &= 2 \\ 2x - 2y &= 5 \end{aligned}$$

   b.
   $$\begin{aligned} 3x - 9y &= 6 \\ x &= 3y + 2 \end{aligned}$$

18.9.2 Nonlinear systems of equations

In this section we will consider some examples of nonlinear systems of equations, to illustrate methods of solutions and some general phenomena. Other examples can be found in parts I, and II.

As mentioned in the previous section, whereas liner systems of equations have either 0, 1, or an infinite number of solutions, nonlinear systems may have, for instance, two or three solutions. Here is an example:

$$\begin{aligned} x\,(x-3) + 2x\cdot y &= 0 \\ 3x + y &= 7 \end{aligned} \tag{18.65}$$

The first equation has two possible solutions: $x = 0$ or $x - 3 + 2y = 0$. Filling in $x = 0$ in the second equation gives $y = 7$, so one solution is the combination $x = 0, y = 7$. To find the other solution, we must solve the linear system:

The first equation can be written as $x\,(x - 3 + 2y) = 0$.

$$\begin{aligned} x - 3 + 2y &= 0 \\ 3x + y &= 7 \end{aligned} \tag{18.66}$$

Using the methods of the previous section, we find: $x = 11/5, y = 2/5$.

Here is an example of a system with four solutions:

$$\begin{aligned} y^2 + 10x\cdot y &= 2y\,(1-y) \\ x\,(x-2) + x\cdot y &= x \end{aligned} \tag{18.67}$$

The first equation is satisfied if $y = 0$ or $y + 10x = 2(1-y) \Rightarrow 3y + 10x - 2 = 0$. The second equation is satisfied if: $x = 0$ or $x - 2 + y = 1 \Rightarrow x + y - 3 = 0$. There are four possible combinations of x and y that satisfy both equations:

$$\begin{aligned} &x = 0 \text{ and } y = 0 \\ &y = 0 \text{ and } x + y = 3 \Rightarrow x = 3 \\ &x = 0 \text{ and } 3y + 10x - 2 = 0 \Rightarrow y = \frac{2}{3} \\ &x + y = 3 \text{ and } 3y + 10x - 2 = 0 \Rightarrow x = 1, y = 4 \end{aligned} \tag{18.68}$$

Exercises and assignments

1. Find all solutions of the following system of equations.

$$\begin{aligned} x\cdot(x-1) - 3x\cdot y &= 0 \\ x + y &= 1 \end{aligned}$$

2. Find all solutions of the following system of equations.

$$\begin{aligned} \frac{x\cdot y}{x+2} - 5y &= y \\ y - \frac{x\cdot y}{x+2} &= 0 \end{aligned}$$

19
Differentiation

The *difference quotient* of a function $f(x)$ on an interval $[a, b)$ is defined as:

$$\frac{f(b) - f(a)}{b - a} \tag{19.1}$$

It corresponds to the slope of the line that connects the points $(a, f(a))$ and $(b, f(b))$. If the function $f(x)$ is linear, this value is the same for every choice of a and b, but in nonlinear functions, such as for instance, $\ln(x)$, it is not. If we decrease the interval $[a, b)$, by choosing b closer to a we get an estimate of the rate of change in the point a. The length of the interval is usually called h and the difference quotient on the interval $[x, x+h)$ is written as

Only in linear functions the slope is constant and does not change with the value of x.

$$\frac{f(x+h) - f(x)}{h} \tag{19.2}$$

For small values of h it is a measure of how sensitive $f(x)$ is to changes in x. If we take the limit for h as it approaches zero (from above), we acquire a *differential quotient* :

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h} \tag{19.3}$$

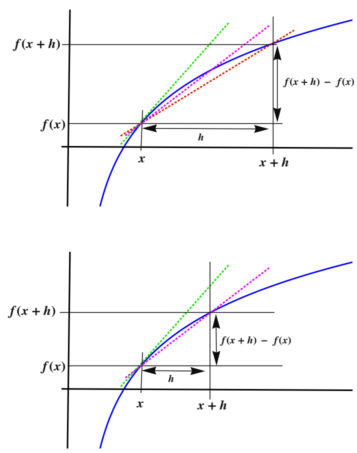


Figure 19.1: The difference quotient and its limit.

The notation for the differential quotient is $\frac{\mathrm{d}f}{\mathrm{d}x}$. Figure 19.1 illustrates how the differential quotient is derived from the difference quotient in a particular point x.

As the value of h decreases, the difference quotient converges to the slope of the tangent to $f(x)$ at the point x. The differential quotient in a point x, therefore, corresponds to the value of this slope. Since, in most cases, this value will change with x, it can be considered as a function of x. This function is called the *derivative* of $f(x)$, and it is usually denoted by $f'(x)$. Since the derivative describes the change in a function, it provides information on where a function is increasing, decreasing, or for which x-values it has an extremum or inflection point. This is especially useful when there are parameters that may affect the shape of a function.

We will use $f'(x)$ to denote the derivative of a function $f(x)$.

19.1 Computation of a derivative

The derivative of basic functions may be determined by calculating the limit in (19.3), for example, for a linear function:

$$
\begin{aligned}
& f(x) = a \cdot x \\
& f(x+h) = a \cdot (x+h) \\
& \tfrac{f(x+h)-f(x)}{h} = \tfrac{a\cdot(x+h)-a\cdot x}{h} = \tfrac{a\cdot h}{h} = a \\
& \lim_{h \downarrow 0} \tfrac{f(x+h)-f(x)}{h} = a
\end{aligned}
\tag{19.4}
$$

In principle, all derivatives may be calculated in this way, but it is not necessary to do so, since the derivatives of basic functions are known (they are listed in Table 19.1) and derivatives of more complex function may be computed from these by application of a limited set of rules.

function $f(x)$	**derivative** $f'(x)$
c	0
x^a, $a \neq 0$	$a \cdot x^{a-1}$
$\ln(x)$	$\frac{1}{x}$
e^x	e^x
a^x	$ln(a) \cdot (a^x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{cos(x)^2}$

Table 19.1: Derivatives of basic functions

We will now briefly summarise the basic rules that can be used to determine derivatives of more complicated functions, and give some examples.

Linearity rule

$$(c \cdot f(x))' = c \cdot f'(x) \tag{19.5}$$

Sum rule

$$(f(x) + g(x))' = f'(x) + g'(x) \tag{19.6}$$

Example:

$$
\begin{aligned}
h(x) &= 2x^2 + \ln(x) \\
h'(x) &= 4x + \tfrac{1}{x}
\end{aligned}
\tag{19.7}
$$

Product rule

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + g'(x) \cdot f(x) \tag{19.8}$$

Example:

$$
\begin{aligned}
h(x) &= x^2 \cdot e^x \\
h'(x) &= 2x \cdot e^x + x^2 \cdot e^x
\end{aligned}
\tag{19.9}
$$

Quotient rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - g'(x)\cdot f(x)}{(g(x))^2} \tag{19.10}$$

Example:

$$\begin{aligned} h(x) &= \tfrac{x^2+1}{4x-3} \\ h'(x) &= \tfrac{2x\cdot(4x-3)-4\cdot\left(x^2+1\right)}{(4x-3)^2} \end{aligned} \tag{19.11}$$

Chain rule

$$\begin{aligned} &\text{If } h(x) = f(g(x)), \\ &\text{then} \\ &h'(x) = f'(g(x))\cdot g'(x) \end{aligned} \tag{19.12}$$

Example:

$$\begin{aligned} &h(x) = e^{ax} \\ &f(g) = e^{g} \Rightarrow f'(g) = e^{g} = e^{ax} \\ &g(x) = ax \Rightarrow g'(x) = a \\ &h'(x) = a\cdot e^{ax} \end{aligned} \tag{19.13}$$

Example:

$$\begin{aligned} &h(x) = \tfrac{1}{\ln(x)} \\ &f(g) = \tfrac{1}{g} \Rightarrow f'(g) = \tfrac{-1}{g^2} = \tfrac{-1}{\ln(x)^2} \\ &g(x) = \ln(x) \Rightarrow g'(x) = \tfrac{1}{x} \\ &h'(x) = \tfrac{-1}{(\ln(x))^2}\cdot\tfrac{1}{x} = -\tfrac{1}{x(\ln(x))^2} \end{aligned} \tag{19.14}$$

Exercises and assignments

1. Compute the derivatives of the following functions:

   a. $f(x) = e^{x^2}$

   b. $f(x) = (x-2)(2x+3)$

   c. $f(x) = \sin(3x)$

   d. $f(x) = 3x\cos(5x)$

   e. $f(x) = \dfrac{x+1}{x^2-2}$

   f. $f(x) = e^x(2x^3 - x^2)$

2. Compute the derivatives of the following functions:

   a. $f(x) = \alpha\cdot x^2 + \beta\cdot x$

   b. $f(x) = 3e^{a\cdot x}$

   c. $f(x) = e^{a\cdot x+b}$

   d. $f(x) = a\cdot x\cdot e^{b\cdot x}$

19.2 *Second derivatives and the shape of functions*

The derivative $f'(x)$ is called the *first derivative* of $f(x)$. The *second derivative* of $f(x)$ is the derivative of $f'(x)$. It is denoted by $f''(x)$. The same computational rules apply. For example:

$$\begin{aligned} f(x) &= 3x^2 + 4x \\ f'(x) &= 6x + 4 \\ f''(x) &= 6 \end{aligned} \tag{19.15}$$

In the same way third, and higher order derivatives may be computed.

The second derivative describes the change in the first derivative. For instance, if $f(t)$ is the location of an automobile at time t, then $f'(t)$ would describe its velocity at that time, and $f''(t)$ its acceleration. In terms of the shape of functions, the second derivative determines whether a function is *concave* or *convex*.

If the second derivative is positive, the value of the first derivative of $f(x)$ increases with x. If the function increases, this implies that the slopes of the tangents to the function become larger as x increases. If the function decreases, the slopes of the tangents to the function will become less negative. In this case the shape of the function is called *convex*. If any two points on a convex curve are connected by a line segment, the segment will lie above the curve. Figure 19.2 shows examples of respectively an increasing and a decreasing convex function. For the increasing function (top figure) the first derivative is positive, whereas for the decreasing function (bottom) it is negative. Since both functions are convex, however, their second derivatives are both negative. An example of an increasing convex function is $f(x) = e^x$, and an example of a decreasing one is $f(x) = e^{-x}$. Functions need not be monotonically increasing or decreasing to be convex, for instance $f(x) = x^2$ is a non-monotonically changing, convex function.

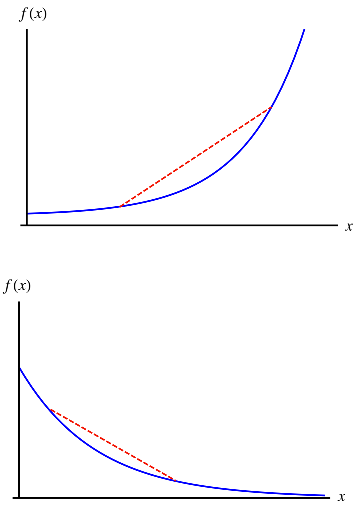


Figure 19.2: Examples of an increasing (top) and a decreasing (bottom) convex function. A line segment connecting any two points on a convex curve will lie above the curve.

If the second derivative is negative, the value of the first derivative of $f(x)$ decreases with x, and the shape of the function is called *concave*. If any two points on a concave curve are connected by a line segment, the segment will lie below the curve. Figure 19.3 shows examples of respectively an increasing and a decreasing concave function. An example of an increasing concave function is $f(x) = \log(x)$, and an example of a decreasing one is $f(x) = \log(-x), (x < 0)$. An example of a concave, non-monotonic function is $f(x) = -x^2$.

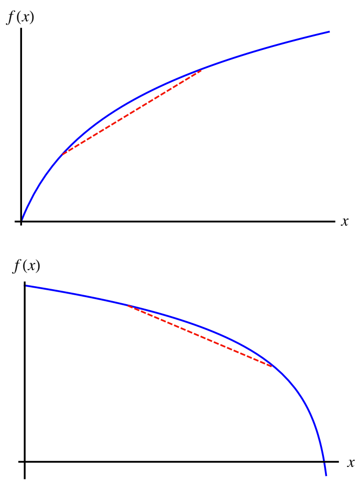


Figure 19.3: Examples of an increasing (top) and a decreasing (bottom) concave function. A line segment connecting any two points on a concave curve will lie below the curve.

Note that functions need not be convex or concave for all values of x in their domain. For instance, the function $f(x) = x^3$ is concave when $x < 0$ and convex when $x > 0$. At the point $x = 0$ the shape of the function changes from concave to convex. Such a point is called an *inflection point*.

The local shape of a function in a point $x = a$ can be determined from the values of the first and second derivatives in that point. If, for instance the first derivative is zero, the tangent to the function in the point a is horizontal. If, in addition, the second derivative is negative, the function is convex at that point. From these two characteristics, if can be inferred that the function has a minimum at this point. If the second derivative would be positive, the point would be a maximum. Figure 19.4 gives an overview of how the first and second derivatives at a point $x = a$ together determine the shape of a function at that point.

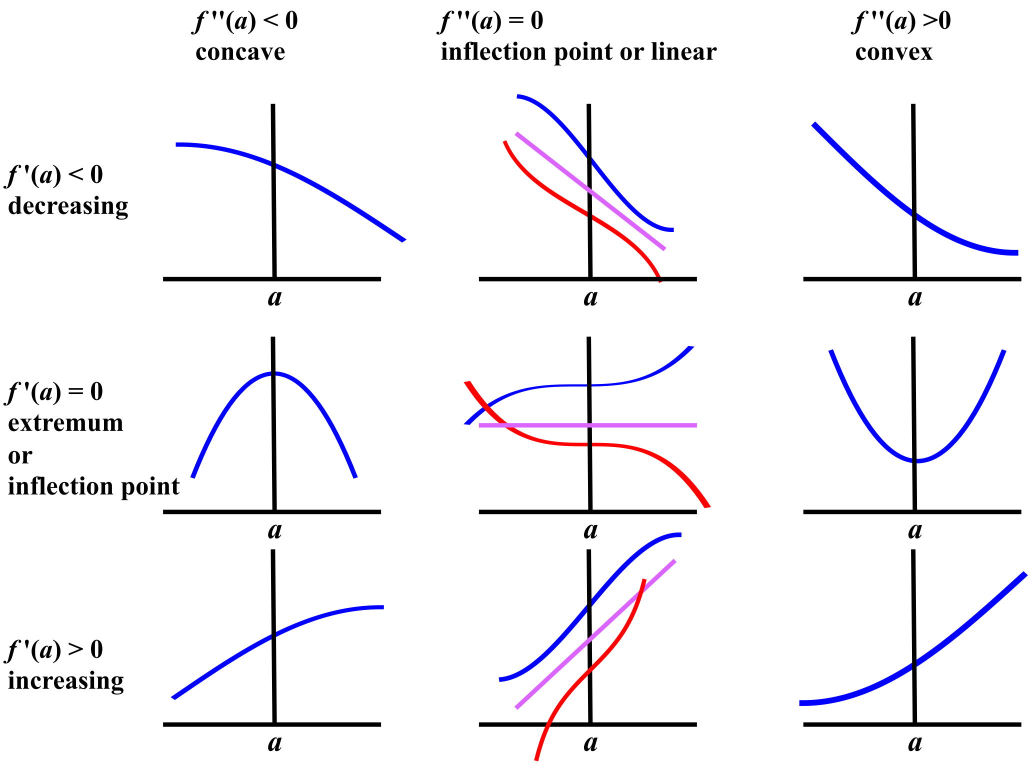


Figure 19.4: Relation between a function's shape and its first two derivatives.

Exercises and assignments

1. Match the graph of each function in (a)-(d) of Fig. 19.5 with the graph of its derivative in I-IV. Motivate your choices.

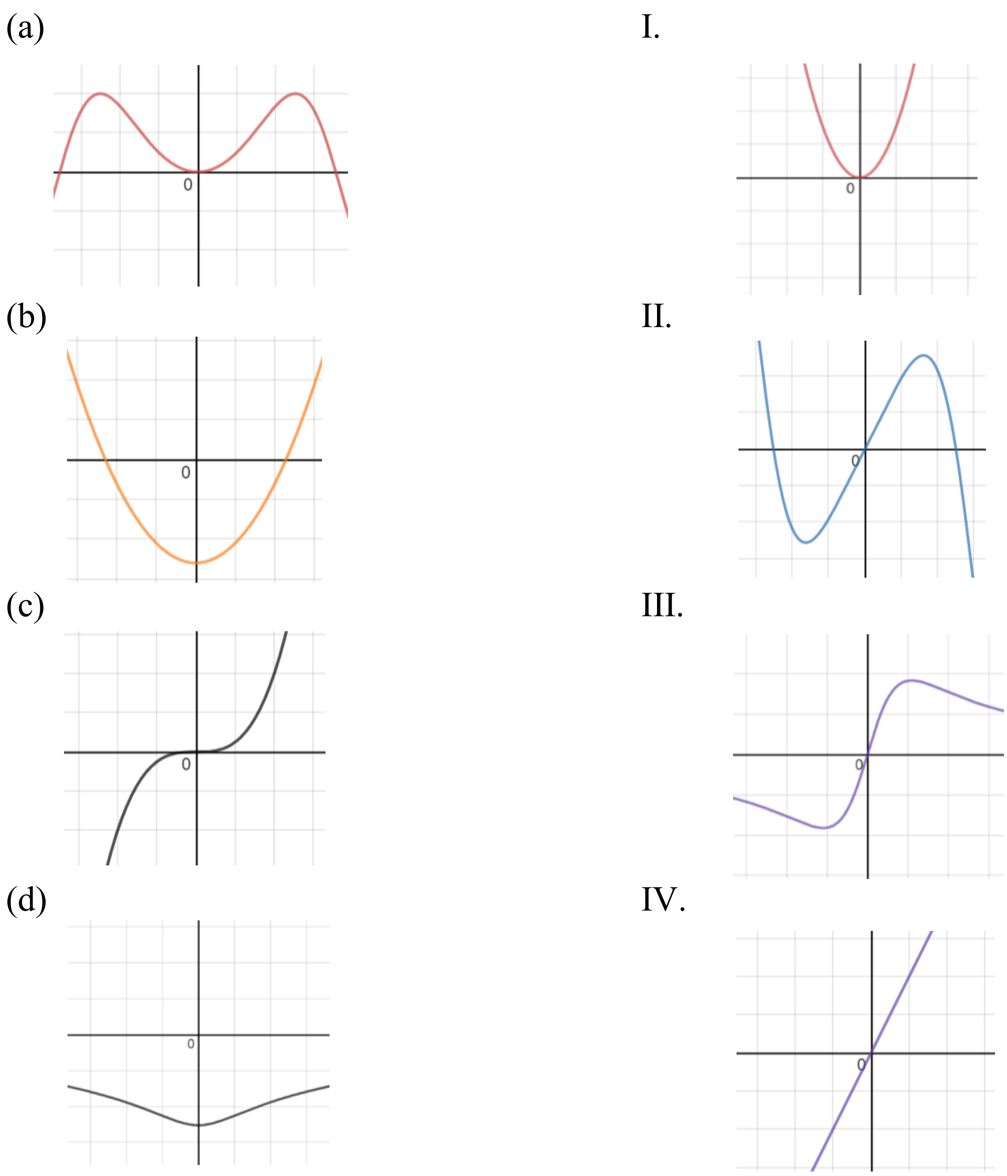


Figure 19.5: Exercise 1

2. Compute the first and second derivatives of the following functions:

   a. $f(x) = 4x - 3x^2$
   b. $f(x) = \sqrt{x}$
   c. $f(x) = \frac{3}{x}$
   d. $f(x) = 3^x$
   e. $f(x) = 6x^2 - 3x + 2$

f. $f(x) = x^{-2}$
g. $f(x) = x^{\frac{3}{2}}$

19.3 Partial derivatives

In the previous sections, we have considered derivatives of *univariate functions*, that depend on a single variable x. Here, we will look at *multivariate functions*, that depend on several variables $x_1, \ldots, x_n$. An example of a two-variate function is:

$$f(x_1, x_2) = {x_1}^2 - 2 \cdot x_1 \cdot x_2 + {x_2}^3 \tag{19.16}$$

For a multivariate function, we cannot simply speak about a 'derivative': since there are multiple variables, we need to specify to which of them the derivative is taken. The *partial derivative* is the derivative with respect to one of the variables, with the others held constant. In other words: the other variables are treated as constants when the derivative is taken. To distinguish partial from total derivatives, we use the following notation for the partial derivative of f to a variable x_i:

$$\frac{\partial f}{\partial x_i} \tag{19.17}$$

For instance, the partial derivatives of the function in (19.16) are:

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 2x_1 - 2x_2 \\ \frac{\partial f}{\partial x_2} &= -2x_1 + 3{x_2}^2 \end{aligned} \tag{19.18}$$

Exercises and assignments

1. Compute the partial derivatives to x and to y for the following functions:

   a. $f(x, y) = 2x \cdot y + 3x - y$
   b. $f(x, y) = a \cdot x^2 + b \cdot y^2$
   c. $f(x, y) = x \cdot e^y$

19.4 Approximation

The technical background for this section is given in chapters 19 and 17.

An important tool in the analysis of nonlinear models is approximation of functions by polynomials, so-called *Taylor approximation*. In the univariate case, this means that $f(x)$ is replaced by a line, parabola, or higher order function. But Taylor approximations can also be applied in higher dimensions, for instance, a plane, or a quadratic surface, may be used to approximate a two-dimensional function.

19.4.1 Approximation of univariate functions

As an example of a Taylor approximation, consider the following, nonlinear function:

$$f(x) = \ln\left(x^2 + 1\right) \tag{19.19}$$

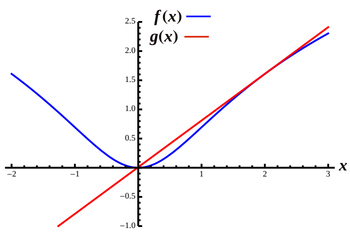


Figure 19.6: Graph of the function in (19.19) and its first order Taylor approximation in he point $x = 2$.

The *first order Taylor approximation* in the point $x = 2$ equals:

$$g(x) = \ln(5) + \frac{4}{5} \cdot (x-2) \tag{19.20}$$

Often, the indication 'Taylor' is dropped, and we simply refer to the *first order approximation*.

The graph of the function $g(x)$ corresponds to the tangent line to the function $f(x)$ in the point $x = 2$. This is illustrated in Fig. 19.6. You can see that this is indeed the equation of that line as follows:

- $g(2) = \ln(5) = f(2)$, so $g(x)$ indeed goes to the point $2, f(2)$
- $g'(2) = \frac{4}{5}$, and, as shown below, this equals the value of $f'(2)$. Therefore, $g(x)$ is indeed tangent to the graph of $f(x)$ in this point.

As you can see in the figure, $g(x)$ approximates $f(x)$ reasonably well as long as x is close to 2. At x-values far removed from this point the approximation does not work. Therefore, we always speak of 'approximation in a point' or 'in the vicinity of a point'. In another point, we need different approximation. For instance, the first order approximation of $f(x)$ in the point $x = -1$ equals:

$$h(x) = \ln(2) - (x+1) \tag{19.21}$$

This is illustrated in Fig. 19.7.

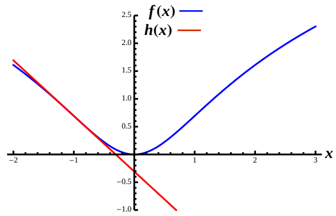


Figure 19.7: Graph of the function in (19.19) and its first order Taylor approximation in he point $x = -1$.

The general formula for the *k-th order Taylor approximation* of a function in a point $x = a$ is:

$$\begin{aligned} f(x) \approx f(a) + f'(a) \cdot (x-a) + \tfrac{1}{2} f''(a) \cdot (x-a)^2 + \ldots \\ + \tfrac{1}{k!} \cdot f^{(k)}(a) \cdot (x-a)^k \end{aligned} \tag{19.22}$$

This is also sometimes called a *Taylor expansion* around the point a.

where $f^{(k)}(a)$ denotes the kth derivative of $f(x)$ in the point $x = a$.

If you fill in $x = a$ in this formula, you indeed get $f(a)$ on both sides. If you take first derivatives, and fill in a you get $f'(a)$ on both sides, if you take second derivatives, you will get $f''(a)$ on both sides, etc. So the approximating polynomial on the right-hand side is chosen is such a way that all derivatives of $f(x)$ up to the kth one are the same as those of the approximating function in the point $x = a$.

To see how this formula is applied, we will show how the approximations in the previous examples are based on the equation in (19.22). The derivative of $f(x)$ in (19.19) equals (applying the chain rule):

$$f'(x) = \frac{2x}{x^2+1} \tag{19.23}$$

So, according to (19.22) the first order approximation in $x = 2$ is:

$$\begin{aligned} f(x) &\approx f(2) + f'(2) \cdot (x-2) \\ &= \ln\left(2^2+1\right) + \frac{2 \cdot 2}{2^2+1} \cdot (x-2) \\ &= \ln(5) + \frac{4}{5} \cdot (x-2) \end{aligned} \tag{19.24}$$

Which gives us indeed the equation of $g(x)$ in (19.20). In a similar way we find the approximation in the point $x = -1$:

$$\begin{aligned}
f(x) &\approx f(-1) + f'(-1) \cdot (x - (-1)) \\
&= \ln\left((-1)^2 + 1\right) + \frac{2 \cdot (-1)}{(-1)^2 + 1} \cdot (x + 1) \\
&= \ln(2) + \frac{-2}{2} \cdot (x + 1) \\
&= \ln(2) - (x + 1) \qquad (19.25)
\end{aligned}$$

Which corresponds to the function $h(x)$ in (19.21).

To acquire the second order Taylor approximation of the function $f(x)$ in the point $x = 2$, we first need to determine the value of the second derivative in that point:

$$\begin{aligned}
f''(x) &= \frac{2 \cdot (x^2 + 1) - (2x) \cdot (2x)}{(x^2 + 1)^2} = \frac{-2x^2 + 2}{(x^2 + 1)^2} \\
f''(2) &= \frac{-2 \cdot 4 + 2}{(4 + 1)^2} = -\frac{6}{25} \qquad (19.26)
\end{aligned}$$

(using the quotient rule for derivatives). As you can see from (19.22) the second order approximation is acquired by adding a term $\frac{1}{2} \cdot f''(a) \cdot (x - a)^2$ to the first order approximation, so:

$$\begin{aligned}
f(x) &\approx \ln(5) + \frac{4}{5} \cdot (x - 2) + \frac{1}{2} \cdot \left(-\frac{6}{25}\right) \cdot (x - 2)^2 \\
&= \ln(5) + \frac{4}{5} \cdot (x - 2) - \frac{3}{25} \cdot (x - 2)^2 \qquad (19.27)
\end{aligned}$$

Figure 19.8 shows the function $f(x)$ with its two approximations in the point $x = 2$. The approximation improves as higher order terms are added, but here we will at most consider second order approximations.

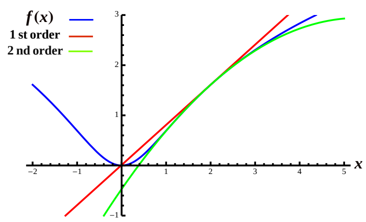


Figure 19.8: Graph of the function in (19.19), with its first(red)- and second order (green) Taylor approximation in the point $x = 2$.

Exercises and assignments

1. What is the second order approximation of the function $f(x)$ in (19.19) in the point $x = -1$?

2. What is the general expression for the second order approximation of the function $f(x)$ in (19.19) in the point $x = a$?

3. Give the first order approximation of the function: $f(x) = e^{x^2}$ in the point $x = 1$

4. Give the second order approximation of the function: $f(x) = e^{x^2}$ in the point $x = 1$

19.4.2 *Approximation of multivariate functions*

In this section we use matrix notation, see 21 and partial derivatives, see 19.

In many applications, several variables are involved, and, as a consequence, models contain multivariate functions. An example of a function that depends on two variables, x_1 and x_2 is:

$$f(x_1, x_2) = 2x_1 \cdot (1 - x_1) - x_1 \cdot x_2 \qquad (19.28)$$

In vector-matrix notation, the first order Taylor approximation of a multivariate function looks very similar to the univariate case:

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \mathbf{f}'(\mathbf{a})\,(\mathbf{x} - \mathbf{a}) \tag{19.29}$$

where $\mathbf{x}$ is the vector of variables $x_1, \ldots, x_n$ and $\mathbf{a}$ the vector of coordinates of the point in which the approximation is made, $a_1, \ldots a_n$. $\mathbf{f}'(\mathbf{a})$ denotes a row vector of the partial derivatives of $f(\mathbf{x})$ with respect to the variables x_i, evaluated in the point $\mathbf{a}$, and $\mathbf{x} - \mathbf{a}$ is a column vector of the differences $x_i - a_i$.

We will only consider the first order approximation of multivariate functions in this book.

For a two-variate function, this equation states:

$$\begin{aligned} f(x_1, x_2) &\approx f(a_1, a_2) + \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)_{a_1, a_2} \cdot \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \end{pmatrix} \\ &= f(a_1, a_2) + \left(\frac{\partial f}{\partial x_1}\right)_{a_1, a_2} (x_1 - a_1) + \left(\frac{\partial f}{\partial x_2}\right)_{a_1, a_2} (x_2 - a_2) \end{aligned} \tag{19.30}$$

The notation $\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)_{a_1, a_2}$ means that the partial derivatives are evaluated at the point a_1, a_2.

As an example, consider the first order approximation of the function in (19.28) in the point $(1, 1)$. Filling in these values for x_1 and x_2 we get:

$$f(1, 1) = 2 \cdot 1 \cdot 0 - 1 \cdot 1 = -1 \tag{19.31}$$

The partial derivatives of the function are:

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 2 - 4x_1 - x_2 \\ \frac{\partial f}{\partial x_2} &= -x_1 \end{aligned} \tag{19.32}$$

In the point $(1, 1)$ this gives:

$$\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)_{1,1} = (2 - 4 - 1, -1) = (-3, -1) \tag{19.33}$$

So, applying (19.30) gives:

$$f(x_1, x_2) \approx -1 - 3\,(x_1 - 1) - (x_2 - 1) \tag{19.34}$$

19.4.3 Approximation of vector-valued functions

In this section we use matrix notation, see 21 and partial derivatives, see 19.

A vector-valued function consists of a vector of functions. We will here consider situations where the number of functions equals the number of variables involved, i.e. for an n-dimensional system we have n variables $x_1, \ldots, x_n$ and the vector-valued function, denoted by $\mathbf{F}(\mathbf{x})$, consists of n functions :

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{pmatrix} \tag{19.35}$$

For example, the following two-dimensional system of functions:

$$\begin{aligned} f_1(x_1, x_2) &= 2x_1 \cdot (1 - x_1) - x_1 \cdot x_2 \\ f_2(x_1, x_2) &= 3x_1 \cdot x_2 - x_2 \end{aligned} \tag{19.36}$$

can be considered as a vector-valued function defined by:

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} 2x_1 \cdot (1 - x_1) - x_1 \cdot x_2 \\ 3x_1 \cdot x_2 - x_2 \end{pmatrix} \tag{19.37}$$

Each of the functions in this vector can be approximated by the method described in the previous subsection (see (19.29)), so the first order approximation of a vector valued function involves no new techniques, only some new notation and terminology.

In vector-matrix notation, the first order approximation in a point $\mathbf{a}$ is:

$$\mathbf{F}(\mathbf{x}) \approx \mathbf{F}(\mathbf{a}) + \mathbf{F}'(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) \tag{19.38}$$

In the two-dimensional case, this equation states:

$$\begin{aligned} \mathbf{F}(\mathbf{x}) &= \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} \\ &\approx \begin{pmatrix} f_1(a_1, a_2) \\ f_2(a_1, a_2) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}_{a_1, a_2} \cdot \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \end{pmatrix} \end{aligned} \tag{19.39}$$

This means that each of the functions $f_i(x_1, x_2)$ is approximated in the way described in Eq. (19.30).

As an example, consider the first order approximation of (19.37) in the point $(1, 1)$.The approximation for f_1 in this example was already derived in the previous subsection, and is given in (19.34). For the second function we find:

$$f_2(1, 1) = 3 - 1 = 2 \tag{19.40}$$

and the partial derivatives are:

$$\begin{aligned} \frac{\partial f_2}{\partial x_1} &= 3x_2 & &\Rightarrow \left(\frac{\partial f_2}{\partial x_1}\right)_{1,1} = 3 \\ \frac{\partial f_2}{\partial x_2} &= 3x_1 - 1 & &\Rightarrow \left(\frac{\partial f_2}{\partial x_2}\right)_{1,1} = 2 \end{aligned} \tag{19.41}$$

so the approximation for f_2 is:

$$\begin{aligned} f_2(x_1, x_2) &\approx f_2(1, 1) + \left(\frac{\partial f_2}{\partial x_1}\right)_{1,1} (x_1 - 1) + \left(\frac{\partial f_2}{\partial x_2}\right)_{1,1} (x_2 - 1) \\ &= 2 + 3(x_1 - 1) + 2(x_2 - 1) \end{aligned} \tag{19.42}$$

Since the approximation of vector-valued functions is essentially the same as the approximation of separate, multivariate functions, you might wonder why we have devoted a separate subsection on this. The reason is that the notation in (19.38) is very useful in the context of multivariate dynamical models. The matrix of partial derivatives, $\mathbf{F}'(\mathbf{a})$ is called the *Jacobian matrix*, or *Jacobian* of the function $\mathbf{F}(\mathbf{x})$. It plays an important role in the analysis of such models.

20

Integration

The inverse of differentiation is called integration. A derivative of a function, $f'(x)$ is given, and we are looking for the original function $f(x)$. Since the derivative of a constant is zero, however, we cannot fully identify the original function, unless additional information is given. For instance: the derivative of $f(x) = 2x$ equals $f'(x) = 2$, but this is also the derivative of $f(x) = 2x + 4$, or $f(x) = 2x + 2$.

When integrating, the function that we start with is denoted by $f(x)$. Suppose that $f(x)$ is the derivative of $F(x)$. Then we call $F(x)$ an *antiderivative* of $f(x)$. The notation for the antiderivative is an integral sign:

Alternative names for an antiderivative are primitive function, primitive integral, or indefinite integral.

$$F(x) = \int f(x)\,\mathrm{d}x \tag{20.1}$$

Since the derivative of a constant is zero, we may add any constant to $F(x)$ and still get the same derivative $f(x)$, so, for any c we have:

$$F(x) + c = \int f(x)\,\mathrm{d}x \tag{20.2}$$

20.1 Finding antiderivatives

Antiderivatives of several basic functions are listed in Table 20.1.

function $f(x)$	antiderivative $\int f(x)\,\mathrm{d}x$
x	$\frac{1}{2} \cdot x^2 + c$
x^2	$\frac{1}{3} \cdot x^3 + c$
$x^n,\ (n \neq -1)$	$\frac{1}{n+1} \cdot x^{n+1} + c$
$\sin(x)$	$-\cos(x) + c$
$\cos(x)$	$\sin(x) + c$
e^{ax}	$\frac{1}{a} \cdot e^{ax} + c$
$a^x,\ (a > 0)$	$\frac{1}{\ln(a)} \cdot a^x + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
$\ln(x)$	$x \cdot \ln(x) - x + c$

Table 20.1: Some basic functions with their antiderivatives. In all cases c stands for an arbitrary constant.

Just as with differentiation, there are certain rules for integration, that can be used to find the antiderivative. We will list them here with some examples.

All these rules can be derived from the rules for differentiation.

Multiplication by a constant
If $F(x)$ is an antiderivative of $f(x)$, then $a \cdot F(x)$ is an antiderivative of $a \cdot f(x)$. Example:

$$\int 4 \cdot x^2 \mathrm{d}x = 4 \cdot \frac{1}{3}x^3 + c \tag{20.3}$$

You may check the validity of this rule if you wish: Differentiation of the right hand side gives the expression in the integral. The same holds for the other rules, given below.

Summation
If $F(x)$ and $G(x)$ are antiderivatives of respectively $f(x)$ and $g(x)$, then $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$. Example:

$$\int \left(x^2 + 3x\right) \mathrm{d}x = \frac{1}{3}x^3 + 3 \cdot \frac{1}{2}x^2 + c \tag{20.4}$$

Addition of a constant to the function argument
If $F(x)$ is an antiderivative of $f(x)$ then $F(x+a)$ is an antiderivative of $f(x+a)$. Example:

$$\int \frac{1}{x+3} \mathrm{d}x = \ln|x+3| + c \tag{20.5}$$

Multiplication of the function argument by a constant
If $F(x)$ is an antiderivative of $f(x)$ then $\frac{1}{a}F(a \cdot x)$ is an antiderivative of $f(a \cdot x)$. Example:

$$\int e^{2x} \mathrm{d}x = \frac{1}{2}e^{2x} + c \tag{20.6}$$

Antiderivatives of given functions may sometimes be found by using the rules listed above, and additional computational 'tricks' that are given in section 20.3. Unlike derivatives, however, it is not always possible to find explicit expressions for antiderivatives. In such cases, numerical procedures may be used to compute approximate values for (definite) integrals.

Exercises and assignments

1. Give the outcomes of the following integrals:

    a. $2x^3 \mathrm{d}x$
    b. $\int \left(x^2 + 4x - 1\right) \mathrm{d}x$
    c. $\int \left(5x^2 + 6\right) \mathrm{d}x$

2. Give the outcomes of the following integrals:

    a. $\int 4e^{3x} \mathrm{d}x$
    b. $\int \frac{2}{x-1} \mathrm{d}x$
    c. $\int \frac{1}{2x+3} \mathrm{d}x$

20.2 Definite integrals

The definition and notation for a definite integral are as follows:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) \tag{20.7}$$

Here is an example:

$$\int_a^b x\mathrm{d}x = \frac{1}{2}b^2 - \frac{1}{2}a^2 \tag{20.8}$$

Note that, if values of the boundaries a and b are given, the outcome is a number, not a function, as would be the case with an indefinite integral. For example, the integral from 1 to 2 of the function x is found by filling in these values in the outcome, and equals 1.5.

The outcome of an indefinite integral is, in fact, a collection of functions, since c is arbitrary.

For convenience, we will introduce a new notation for the outcome of a definite integral:

$$[F(x)]_a^b = F(b) - F(a) \tag{20.9}$$

Using this notation, for example, (20.8) is written as:

$$\int_a^b x\mathrm{d}x = \left[\frac{1}{2}x^2\right]_a^b = \frac{1}{2}b^2 - \frac{1}{2}a^2 \tag{20.10}$$

It can be shown that the definite integral is the surface between a graph of the integrated function and the horizontal axis. This is demonstrated for the example in (20.8) in Fig. 20.1. In this particular example, the truth of the assertion is easily checked: the surface under the graph is equal to half the surface of a (2 by 2) square minus half the surface of a (1 by 1) square.

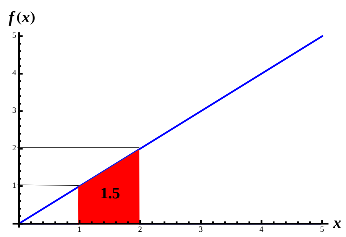


Figure 20.1: The integral from 1 to 2 of the function $f(x) = x$ equals the surface under the function, which is 1.5.

If the function is negative, and lies beneath the x-axis the integral becomes negative too. Its the absolute value still corresponds to the surface between the function and the x-axis. For instance, from (20.8) we can derive that the integral from $a = -2$ to $b = -1$ of x equals -1.5. This is exactly minus times the value that we found before. Because of the symmetry, you can easily see that the surfaces are equal (see Fig. 20.2). In the next subsection the general validity of the relationship between definite integrals and areas under a curve is shown.

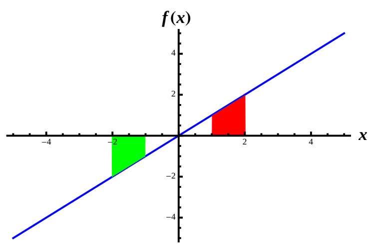


Figure 20.2: The integral from -2 to -1 of the function $f(x) = x$ equals minus the surface between the function and the horizontal axis. Since the surfaces of the green and red areas are the same, the integral equals -1.5.

Exercises and assignments

1. Give the outcomes of the following integrals:

   a. $\int_2^3 (2x+5)\,\mathrm{d}x$

   b. $\int_1^5 \frac{1}{2x}\mathrm{d}x$

   c. $\int_0^4 2e^{5x}\mathrm{d}x$

20.2.1 *The fundamental theorem of calculus*

The fundamental theorem of calculus states that the definite integral is indeed the surface between the curve of a function and the x-axis, as stated above. In this subsection we will show that this is indeed true. We will do this by showing that, when $F(a) - F(0)$ is the area under a curve of a function $f(x)$ between $x = 0$ and $x = a$, then the derivative of $F(x)$ in the point a equals $f(a)$, in other words $f(a) = F'(a)$. Since this holds for arbitrary values of a, it follows that $F(x)$ is indeed the antiderivative of $f(x)$.

Consider a function $f(x)$, and let $F(a) - F(0)$ be the area under the curve between 0 and a a point $x = a$, as illustrated in Fig. 20.3. Now consider the area $F(a+h) - F(0)$, up to a point further to the right, $a + h$. As illustrated in Fig. 20.3, this area is nearly equal to the original, $F(a) - F(0)$ plus the area of a rectangle of width h and height $f(a)$, so:

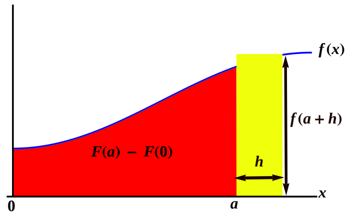


Figure 20.3: A function $f(x)$ with its area under the curve from $x = 0$ to $x = a$. Further explanation: see main text.

This reasoning is not (yet) a formal proof, since we have only considered a situation where $f(x)$ and a are positive. The reasoning can, however, be straightforwardly extended to situations where this is not so.

$$F(a+h) - F(0) \approx F(a) - F(0) + h \cdot f(a) \tag{20.11}$$

This equation can be rearranged as follows:

$$\begin{aligned}
F(a+h) - F(0) &\approx F(a) - F(0) + h \cdot f(a) && \\
F(a+h) &\approx F(a) + h \cdot f(a) && (\text{add } F(0) \text{ on both sides}) \\
F(a+h) - F(a) &\approx h \cdot f(a) && (\text{subtract } F(a) \text{ on both sides}) \\
\frac{F(a+h) - F(a)}{h} &\approx f(a) && (\text{divide by } h \text{ on both sides})
\end{aligned} \tag{20.12}$$

If we let h go to zero, we get:

$$\lim_{h \downarrow 0} \frac{F(a+h) - F(a)}{h} = f(a) \tag{20.13}$$

the expression on the left-hand side is the derivative of $F(x)$ in the point a, so:

$$F'(a) = f(a) \tag{20.14}$$

and, since this holds in any point a, we can conclude that $F(x)$ is indeed the antiderivative of $f(x)$.

20.3 *Computation of integrals*

There are several ways that may help to find explicit expressions for antiderivatives, and thus allow you to compute the values of definite integrals. These methods do not always lead to a solution, however. Sometimes it is just not possible to compute an integral exactly, and we may need to resolve to numerical computation methods (see section 20.4).

20.3.1 *Partial fractions*

The following integral cannot be computed directly with the methods that we have considered up to now:

$$\int_1^3 \frac{x}{x+1} \mathrm{d}x \tag{20.15}$$

The reason is that x occurs in the numerator as well as the denominator of the fraction. There is a way, however, to overcome this, and rewrite the fraction in such a way that x only occurs in the denominator:

$$\frac{x}{x+1} = \frac{x+1-1}{x+1} = \frac{x+1}{x+1} - \frac{1}{x+1} = 1 - \frac{1}{x+1} \tag{20.16}$$

As a result, we are able to compute the integral in (20.15):

$$\begin{aligned}
\int_1^3 \frac{x}{x+1} \mathrm{d}x &= \int_1^3 \left(1 - \frac{1}{x+1}\right) \mathrm{d}x \\
&= \int_1^3 1 \mathrm{d}x - \int_1^3 \frac{1}{x+1} \mathrm{d}x \\
&= [x]_1^3 - [\ln(x+1)]_1^3 \\
&= (3-1) - (\ln(4) - \ln(1)) \\
&= 2 - \ln(4),
\end{aligned} \tag{20.17}$$

where we use the notation that was introduced in (20.9).

We will now explain how this method works in general. The original fraction is denoted as:

$$\frac{P(x)}{Q(x)}, \tag{20.18}$$

where $P(x)$ and $Q(x)$ are two different polynomials (see 17.1). The goal is to find a relationship of the form:

$$\frac{P(x)}{Q(x)} = \mathrm{frac}_1 + \mathrm{frac}_2 + \ldots \mathrm{frac}_m, \tag{20.19}$$

where the expressions on the right-hand side are fractions that contain x only in the denominator. The way to proceed depends on the combination of the order of degree of the two polynomials $P(x)$ and $Q(x)$.

If $Q(x)$ has a higher degree than $P(x)$

In this case first factorise $Q(x)$ as far as possible. Then there are different possibilities for the fractions on the right hand side of (20.19):

1. For each of the factors of the type $ax + b$ we need a fraction of the form: $\frac{k}{ax+b}$.

2. For each of the factors of the type $(cx+d)^n$ we will need to use a series of fractions:
$$\frac{k_1}{cx+d}+\frac{k_2}{(cx+d)^2}+\dots\frac{k_n}{(cx+d)^n}$$

3. For quadratic expressions that cannot be factorised we need a fraction of the form:
$$\frac{mx+n}{px^2+qx+r}$$

The next step is to equate the original expression to the series of fractions that you have composed in this way. For example:

$$\frac{3x-1}{(2x+1)(x-1)^2}=\frac{a}{2x+1}+\frac{b}{x-1}+\frac{c}{(x-1)^2} \tag{20.20}$$

The constants a, b and c need to be computed next. This is done by multiplying both sides by the expression in the denominator on the left-hand side. This gives:

$$3x-1=a\cdot(x-1)^2+b\cdot(2x+1)\cdot(x-1)+c\cdot(2x+1) \tag{20.21}$$

Working out the expressions on the right-hand side and rearranging gives:

$$\begin{aligned}3x-1&=a\cdot\left(x^2-2x+1\right)+b\cdot\left(2x^2-x-1\right)+c\cdot(2x+1)\\&=x^2\cdot(a+2b)+x\cdot(-2a-b+2c)+(a-b+c)\end{aligned} \tag{20.22}$$

Since expressions on both sides must be equal, we get the following relations for a, b, and c:

$$\begin{aligned}a+2b&=0\\-2a-b+2c&=3\\a-b+c&=1\end{aligned} \tag{20.23}$$

Solving this system of linear equations gives (see 18.9):

$$a=-\frac{2}{9},b=\frac{1}{9},c=\frac{4}{3} \tag{20.24}$$

Finally, substitution in (20.20) gives:

$$\frac{3x-1}{(2x+1)(x-1)^2}=-\frac{2}{9}\cdot\frac{1}{2x+1}+\frac{1}{9}\cdot\frac{1}{x-1}+\frac{4}{3}\cdot\frac{1}{(x-1)^2} \tag{20.25}$$

If $Q(x)$ has a lower degree than $P(x)$

In this case we first divide $P(x)$ by $Q(x)$, by long division. For example:

$$\frac{x^3+1}{x^2-x} \tag{20.26}$$

Carrying out long division gives:

$$
\begin{aligned}
&x \cdot \left(x^2 - x\right) = x^3 - x^2 && \text{(first part } = \text{x)}\\
&x^3 + 1 - \left(x^3 - x^2\right) = 1 + x^2 = x^2 + 1 && \\
&1 \cdot \left(x^2 - x\right) = x^2 - x && \text{(second part } = 1)\\
&x^2 + 1 - \left(x^2 - x\right) = 1 + x && \text{(left over)}\\
&\text{so} && \\
&\frac{x^3+1}{x^2-x} = x + 1 + \frac{x+1}{x^2-x} && (20.27)
\end{aligned}
$$

The rest term is a fraction with a higher degree polynomial in the denominator than in the numerator, which can be processed in the way that was explained before.

***If** $Q(x)$ **has a the same degree as** $P(x)$*
This type of fractions can be written in the following way:

$$\frac{P(x)}{Q(x)} = c \cdot \frac{x^k}{Q(x)} + \frac{Z(x)}{Q(x)} \tag{20.28}$$

where c is a constant and k is the degree of $Q(x)$ and $Z(x)$ is a polynomial of a degree lower than $Q(x)$.For example:

$$\frac{3x^2+x+1}{x^2+2x+1} = \frac{3x^2}{x^2+2x+1} + \frac{x+1}{x^2+2x+1} \tag{20.29}$$

It was explained above how the second term can be dealt with. To deal with the first expression, use long division. In this case:

$$
\begin{aligned}
&3 \cdot \left(x^2 + 2x + 1\right) = 3x^2 + 6x + 3 && \text{(first part } = 3)\\
&3x^2 - \left(3x^2 + 6x + 3\right) = -(6x + 3) && \text{(left over)}\\
&\text{so} && \\
&\frac{3x^2}{x^2+2x+1} = 3 - \frac{6x+3}{x^2+2x+1} && (20.30)
\end{aligned}
$$

Rewriting a fraction in an integral with the methods explained above usually leads to sums of integrals that can be solved straightforwardly. In some cases additional methods, such as explained in the following subsections may be needed.

20.3.2 *Substitution*

Substitution is a method for solving integrals that is based on the chain rule for differentiation (see (19.12)). The chain rule can be written in a slightly different way, using differential quotients (cf. (19.3)):

$$\frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = \frac{\mathrm{d}f(g)}{\mathrm{d}g} \cdot \frac{\mathrm{d}g(x)}{\mathrm{d}x} \tag{20.31}$$

Taking integrals on both sides and rearranging gives:

$$\begin{aligned}\int \frac{\mathrm{d}f(g(x))}{\mathrm{d}x}\mathrm{d}x &= \int \frac{\mathrm{d}f(g)}{\mathrm{d}g}\cdot\frac{\mathrm{d}g(x)}{\mathrm{d}x}\mathrm{d}x \\ \int \mathrm{d}f(g(x)) &= \int \frac{\mathrm{d}f(g)}{\mathrm{d}g}\mathrm{d}g(x) \\ f(g(x)) &= \int \frac{\mathrm{d}f(g)}{\mathrm{d}g}\mathrm{d}g(x)\end{aligned} \tag{20.32}$$

Another way of writing the result is:

$$\int f'(g(x))\,\mathrm{d}x = f(g(x)) \tag{20.33}$$

Formally a constant should be added because we are dealing with an indefinite integral.

As an example of an application of this rule, consider:

$$\int \frac{x}{x^2+1}\mathrm{d}x \tag{20.34}$$

In this case, let $g(x) = x^2$, so $g'(x) = 2x$. Then (20.34) can be rewritten as:

$$\int \frac{\frac{1}{2}g'(x)}{g(x)+1}\mathrm{d}x = \int \frac{1}{2}\frac{1}{g(x)+1}g'(x)\,\mathrm{d}x \tag{20.35}$$

This gives an equation of the form in (20.33), with

$$f'(g(x)) = \frac{1}{2}\cdot\frac{1}{g(x)+1} \tag{20.36}$$

and, using the methods explained in section 20.1, we find:

$$f(g(x)) = \frac{1}{2}\cdot\ln|g(x)+1| \tag{20.37}$$

filling in the expression for $g(x)$ gives us the solution of (20.34):

$$\int \frac{x}{x^2+1}\mathrm{d}x = \frac{1}{2}\cdot\ln\left(x^2+1\right) \tag{20.38}$$

Since x^2+1 is always positive we do not need to use the 'absolute value' signs in the expression for the logarithm anymore. You may check the validity of the outcome by differentiating the solution.

Exercises and assignments

1. Compute the following integrals, using substitution:
   a. $\int \frac{\ln(t)}{t}\mathrm{d}t$
   b. $\int_0^{\pi} \sin(x)\cdot\cos(x)\,\mathrm{d}x$
   c. $\int_0^{1} x\cdot e^{x^2}\mathrm{d}x$

20.3.3 Partial integration

Partial integration is a method to determine the integral of a product of two functions. It is based on the product rule for differentiation (see (19.8)). Rearranging that equation gives:

$$\begin{aligned}(f(x)\cdot g(x))' &= f(x)\cdot g'(x) + g(x)\cdot f'(x) \\ f(x)\cdot g'(x) &= (f(x)\cdot g(x))' - f'(x)\cdot g(x)\end{aligned} \tag{20.39}$$

integration on both sides gives:

$$\begin{aligned}\int f(x) \cdot g'(x)\,\mathrm{d}x &= \int \left(f(x) \cdot g(x)\right)'\mathrm{d}x - \int g(x) \cdot f'(x)\,\mathrm{d}x \\ &= f(x) \cdot g(x) - \int f'(x) \cdot g(x)\,\mathrm{d}x\end{aligned} \tag{20.40}$$

As an example of applying this rule consider:

$$\int x \sin(x)\mathrm{d}x \tag{20.41}$$

Let $f(x) = x$ and $g'(x) = \sin(x)$,so $f'(x) = 1$ and $g(x) = -\cos(x)$ (see Table 20.1), and:

$$\begin{aligned}\int x \sin(x)\mathrm{d}x &= x \cdot (-\cos(x)) - \int 1 \cdot (-\cos(x))\,\mathrm{d}x \\ &= x \cdot (-\cos(x)) + \int \cos(x)\,\mathrm{d}x \\ &= -x \cdot \cos(x) - \sin(x)\end{aligned} \tag{20.42}$$

Formally a constant should be added because we are dealing with an indefinite integral.

Exercises and assignments

1. Compute the following integrals, using partial integration:
   a. $\int x \cdot e^x \mathrm{d}x$
   b. $\int x \cdot \ln(x)\,\mathrm{d}x$

20.4 *Numerical integration*

It is not always possible to find an explicit expression for an integral. In such situations, computer programs may be used to find an (approximate) answer. This procedure is called numerical integration. In the course of time, very sophisticated methods for numerical integration have been developed, and computer languages such as R contain packages to carry them out, so you won't need to program them yourself. This section only considers the original, relatively simple method that was proposed by Euler. The aim is to give you an impression of what numerical integration entails, and understand what the computer programs are doing. This will help you make more efficient use of such programs, and interpret error messages (should they occur).

As stated above, a definite integral from a to b represents the surface of the area under a graph of a function $f(x)$ (see Fig. 20.3). This surface can be approximated by the sum of the surfaces of rectangles, of width h and heights determined by the values of $f(x)$, as illustrated in Fig. 20.4. As h decreases, the approximation becomes more accurate. This is, essentially, the method that is used in Euler integration.

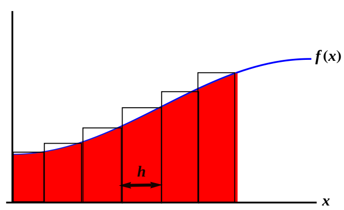


Figure 20.4: Approximation of the surface under the curve.

For example, consider the following integral. Since its value is known, we may use it to illustrate the convergence of the approximation to the real value of the integral, as h decreases.

$$\int_0^1 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3} \tag{20.43}$$

but as an illustration let's see how it can be approximated by numerical integration. If we divide the interval by 2, and take steps of size $h = 1/2$, the approximation becomes:

$$\begin{aligned} &\int_0^1 f(x)\,\mathrm{d}x \approx 0 + \frac{1}{2}\cdot f\left(\frac{1}{2}\right) + \frac{1}{2}\cdot f(1) \\ &\text{so :} \\ &\int_0^1 x^2 \mathrm{d}x \approx 0 + \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{2}\cdot(1)^2 = \frac{1}{8} + \frac{1}{2} = \frac{5}{8} \end{aligned} \tag{20.44}$$

This is not yet a good approximation, so let's decrease the interval, and take $h = 1/4$

$$\begin{aligned} &\int_0^1 f(x)\,\mathrm{d}x \approx 0 + \frac{1}{4}\cdot f\left(\frac{1}{4}\right) + \frac{1}{4}\cdot f\left(\frac{2}{4}\right) + \frac{1}{4}\cdot f\left(\frac{3}{4}\right) + \frac{1}{4}\cdot f(1) \\ &\text{so} \\ &\int_0^1 x^2 \mathrm{d}x \approx 0 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot 1 \\ &= \frac{1}{64} + \frac{1}{16} + \frac{9}{64} + \frac{1}{4} = \frac{15}{32} \end{aligned} \tag{20.45}$$

This is better, but not very good yet. Taking $h = 1/16$ gives:

$$\int_0^1 x^2 \mathrm{d}x \approx 0 + \frac{1}{16}\cdot\left(\frac{1}{16}\right)^2 + \frac{1}{16}\cdot\left(\frac{2}{16}\right)^2 + \ldots + \frac{1}{16}\cdot 1 = \frac{187}{512} \approx 0.365 \tag{20.46}$$

As you can see, the approximation is (slowly) converging to the real value, $1/3$, as h gets smaller. For $h = 0.001$ for instance, the approximate value of the numerical integration is 0.334.

Exercises and assignments

1. Write down the approximation of the following integrals, with $h = 1/3$:

   a. $\int_1^2 \ln(x)\,\mathrm{d}x$

   b. $\int_0^1 x^2 e^x \mathrm{d}x$

21
Matrices and vectors

Matrices and vectors provide efficient ways to represent relationships between several variables. They therefore play an important role in multidimensional models and statistical procedures. For instance: the following system of linear equations:

$$\begin{aligned} 2x + 3y &= 1 \\ 4x - 7y &= 6 \end{aligned} \tag{21.1}$$

may also be represented by an equation involving two vectors:

$$\begin{pmatrix} 2x + 3y \\ 4x - 7y \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \tag{21.2}$$

The vector on the left-hand side may be written as a product of a matrix and a vector:

$$\begin{pmatrix} 2x + 3y \\ 4x - 7y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \tag{21.3}$$

Together this results in an alternative representation of the system of equations in (21.1):

$$\begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \tag{21.4}$$

This notation is further explained in the following sections.

21.1 Basic terminology and notations

In the context of multidimensional systems we need to distinguish scalars, vectors and matrices. A single quantity is called a *scalar*. For instance the number 4.3 is a scalar. We may also talk for instance of scalar variables. A *vector* is a series of quantities arranged as a row or a column. An example of a *column vector* is:

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{21.5}$$

If the quantities are arranged in a row we speak of a *row vector* , for instance:

$$(3, 2.5) \tag{21.6}$$

A *matrix* is a series of quantities that is arranged in 'tabular' way, with rows as well as columns. For example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \\ 6 & 8 \end{pmatrix} \tag{21.7}$$

The quantities contained in a vector or matrix are called its *elements*. For instance, the first element of the vector in (21.5) is 1, its second element is 3 etc. The matrix in (21.7) has six elements.

The elements of a vector or matrix are not necessarily numbers. They may be variables, functions, or mathematical expressions.

The *length* of a vector is its number of elements. The *dimensions* of a matrix are its number of rows and columns. For instance, the matrix in (21.7) has dimension 3×2. We call this a '3 by 2 matrix'.

Note that we may consider a column vector of length n as a matrix with dimensions $n \times 1$, and a row vector of the same length as a 1 by n matrix.

We will denote vectors and matrices by straight, bold-face symbols, for instance:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{21.8}$$

Matrices and vectors are denoted by straight boldface symbols.

Vectors are by default column vectors. To indicate a row vector, we will add an accent to the column's name, for instance the 'row version' of the column vector in (21.8) is:

$$\mathbf{v}' = (1, 2) \tag{21.9}$$

Row vectors are indicated by adding an accent

We may refer to the elements of matrices and vectors by means of subscripts, for instance the first element of the second row of the matrix $\mathbf{A}$ in (21.8) and the second element of the vector $\mathbf{v}$ in this equation are respectively:

$$a_{21} = 4,\ v_2 = 2 \tag{21.10}$$

If a matrix has the same number of rows as it has columns, we call it a *square matrix* . A special type of square matrix is the *identity matrix*. The off-diagonal elements of this matrix are all 0, and its diagonal elements are 1. Square matrices are usually indicated by the symbol $\mathbf{I}$ For instance, the 2 by 2 identity matrix is:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{21.11}$$

Sometimes a subscript is used to indicate the dimension of the identity matrix. In this case it would be $\mathbf{I_2}$

The symbol $\mathbf{I}$ represents an identity matrix. An alternative notation, to indicate explicitly that it has n rows and columns, is $\mathbf{I_n}$.

The *transpose* of a matrix is obtained by switching its rows and columns. It is denoted by a superscript, for instance:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}, \mathbf{A^T} = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \tag{21.12}$$

Note that we can also denote row vectors as the transpose of column vectors, for instance $\mathbf{v}'$ in (21.9) is also the transpose, $\mathbf{v^T}$, of $\mathbf{v}$ in (21.8).

21.2 Calculations with matrices and vectors

If a matrix or a vector is multiplied by a scalar, all its elements are multiplied. Matrix addition (or subtraction) consists of simply summing (subtracting) the separate elements of the matrices or vectors involved. Thus, two matrices or vectors can only be added (or subtracted) if they have the same dimensions. Here are some examples:

$$\begin{gathered}
\mathbf{A}=\begin{pmatrix} 2 & 3 & 6 \\ 4 & 1 & 1 \end{pmatrix}, \mathbf{B}=\begin{pmatrix} 1 & 1 & 4 \\ 2 & 3 & 2 \end{pmatrix}, \mathbf{v}=\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{w}=\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \\
\mathbf{A}+\mathbf{B}=\begin{pmatrix} 3 & 4 & 10 \\ 6 & 4 & 3 \end{pmatrix}, \mathbf{v}+\mathbf{w}=\begin{pmatrix} 7 \\ 7 \\ 10 \end{pmatrix} \\
3\cdot\mathbf{B}=\begin{pmatrix} 3 & 3 & 12 \\ 6 & 9 & 6 \end{pmatrix}, \mathbf{v}/2=\begin{pmatrix} 1 \\ 0.5 \\ 1.5 \end{pmatrix}
\end{gathered} \tag{21.13}$$

Matrix multiplication, however, is different from normal multiplication. It is not just a simple multiplication of the elements of both components. To illustrate this consider the example at the beginning of this chapter, in Eq. (21.3). With a slight rearrangement this equation states that:

Matrix multiplication is different from 'normal multiplication'.

$$\begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ 4x-7y \end{pmatrix} \tag{21.14}$$

What happens is that the first element in the first row of the matrix, 2 is multiplied by the first element of the vector, x, giving $2x$. The second element in the first row, 3 is multiplied by the second element of the vector, y, giving $3y$. The sum of these results, $2x+3y$ provides the first element of the resulting vector. Carrying out a similar procedure with the second row of the matrix gives the second element of the vector on the right-hand side of (21.14).

Thus, the multiplication of a 2×2 matrix with a vector of length 2 works out as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}\cdot x_1 + a_{12}\cdot x_2 \\ a_{21}\cdot x_1 + a_{22}\cdot x_2 \end{pmatrix} \tag{21.15}$$

This illustrates how matrix multiplication works. If we call the matrix in (21.15) $\mathbf{A}$ and the vector $\mathbf{x}$, and the resulting vector $\mathbf{b}$, the matrix multiplication may be written as:

$$\mathbf{A}\cdot\mathbf{x}=\mathbf{b} \tag{21.16}$$

Matrix multiplication is quite straightforward, applying the procedure in (21.15) all the time. It does take some time to get used to it, though, and it is easy to make mistakes. In most applications, however, there is no need to carry out these calculations. It suffices to know how the multiplication is defined.

It is important to realise that there is an essential difference between normal multiplication and the multiplication of matrices and vectors: matrix multiplication does not have the commutative property, so the order of multiplication matters:

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A} \tag{21.17}$$

In matrix multiplication the order matters!

Here is an example to illustrate this:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 3 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 2 & 5 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 3 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 6 & 4 \end{pmatrix}$$

Exercises and assignments

1. Carry out the following matrix multiplications:

   a.
   $$\begin{pmatrix} 3 & 8 \\ -6 & 5 \end{pmatrix} \cdot \begin{pmatrix} 78 & 5 \\ 86 & 2 \end{pmatrix}$$

   b.
   $$\begin{pmatrix} 2 & 3 \\ 6 & -5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

   c.
   $$7 \cdot \begin{pmatrix} 4 & 5 & 1 \\ 8 & 9 & 12 \end{pmatrix}$$

   d.
   $$\begin{pmatrix} -2 & -4 \end{pmatrix} \cdot \begin{pmatrix} 5 & 5 \\ 7 & 8 \end{pmatrix}$$

2.

21.3 *Trace and determinant*

The trace and determinant are important characteristics of square matrices. The trace of a matrix is the sum of its diagonal elements. For instance, the trace of the following matrix:

$$\begin{pmatrix} -1 & 3 \\ 4 & 2 \end{pmatrix} \tag{21.18}$$

equals $-1 + 2 = 1$. More generally, for a 2 by 2 matrix $\mathbf{A}$, we have:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \text{Trace} = a + d \tag{21.19}$$

The determinant of a 2 by 2 matrix is the product of its diagonal elements minus the product of the other two elements, for instance, the determinant of the matrix in (21.18) is $-1 \cdot 2 - 4 \cdot 3 = -14$. The general equation is:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \text{Determinant} = a \cdot d - c \cdot b \tag{21.20}$$

A common notation for the determinant is:

$$|\mathbf{A}| = ad - bc, \text{ or :}$$
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{21.21}$$

General expressions for determinants of higher dimensional matrices are more complex, and will not be considered here. They are given in, for instance Caswell's book on matrix models [1]. A $n \times n$ matrix with a determinant that is not equal to zero has *rank n*. This is also called a *full rank* matrix. The determinant is zero when the columns or rows of the matrix are *linearly dependent*. This means that at least one of the columns (or, rows) can be expressed as a linear combination of the others. For instance, the following matrix has a determinant of zero. This can be computed, but it can also be inferred directly from the fact that its second column equals 2 times the first column (or from the fact that the second row is 1/2 times the first one).

$$\mathbf{B} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \tag{21.22}$$

[1] H. Caswell. *Matrix Population Models*. Sinauer Associates, Inc., Sunderland, Massachusetts, 2006

Exercises and assignments

1. Give the trace and the determinant of each of the following matrices:

   a.
   $$\begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix}$$

   b.
   $$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

   c.
   $$\begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix}$$

2. Give the trace and the determinant of each of the following matrices:

   a.
   $$\begin{pmatrix} -2a & b \\ 3 & 0 \end{pmatrix}$$

   b.
   $$\begin{pmatrix} a & 2 \\ 1 & a+1 \end{pmatrix}$$

c.

$$\begin{pmatrix} a & -b \\ a & 3b \end{pmatrix}$$

22

Answers to exercises of Part III

Chapter 16 Summations and products

section 16.1

1. Consider the following sequence $3, 5, 5, 2, 6, 1, 4$.
   a. The individual elements are referred to as x_i, $i = 1, \ldots, 7$. What is the value of x_5?
   answer: $x_5 = 6$
   b. What is the length of the series, n?
   answer: $n = 7$
   c. Calculate $\sum\limits_{i=1}^{n} x_i$
   answer: 26
   d. Calculate $\sum\limits_{i=2}^{n-1} x_i$
   answer: $5 + 5 + 2 + 6 + 1 = 19$
   e. What is the value of $\bar{x}$?
   answer: 26/7
   f. Calculate $\sum\limits_{i=1}^{n} {x_i}^2$
   answer: 116
   g. What is the variance of the data?
   answer: $\frac{116}{7} - \left(\frac{26}{7}\right)^2 \approx 2.78$
2. Show that
$$\sum_{i=1}^{n} (a \cdot x_i)^2 = a^2 \cdot \sum_{i=1}^{n} {x_i}^2$$
   answer:
$$\sum_{i=1}^{n} (a \cdot x_i)^2 = \sum_{i=1}^{n} a^2 \cdot {x_i}^2 = a^2 \cdot \sum_{i=1}^{n} {x_i}^2$$
3. Show that
$$\frac{1}{n} \sum_{i=1}^{n} (x_i - b) = \bar{x} - b$$
   answer:
$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{n} (x_i - b) &= \frac{1}{n} \left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} b \right) \\ &= \frac{1}{n} \sum_{i=1}^{n} x_i - \frac{1}{n} n \cdot b \\ &= \bar{x} - b \end{aligned}$$

section 16.2

1. The individual elements of the following sequences are referred to as $x_i, i = 1, \ldots n$, where n is the sequence length. For each of the sequences calculate $\prod_{i=1}^{n} x_i$.

   a. $1, 4, 1, 5, 10$
   answer: $\prod_{i=1}^{n} x_i = 200$

   b. $31, 21, 7, 0, 3, 16, 10$
   answer: $\prod_{i=1}^{n} x_i = 0$

   c. $2, 2, 2, 2, 1,$
   answer: $\prod_{i=1}^{n} x_i = 2^4 = 16$

Chapter 17 Properties of elementary functions

section 17.1.1

1. Sketch the graphs of the following linear functions. Do this by calculating their intercepts and taking into account whether their slope is positive or negative. Indicate the values of the intercepts on the axes, and no scale, as was done in Fig. 17.1.

   a. $f(x) = 3 + 2x$
   b. $f(x) = -3 + 2x$
   c. $f(x) = 4 - 3x$
   d. $f(x) = -4 - 3x$

   answer:

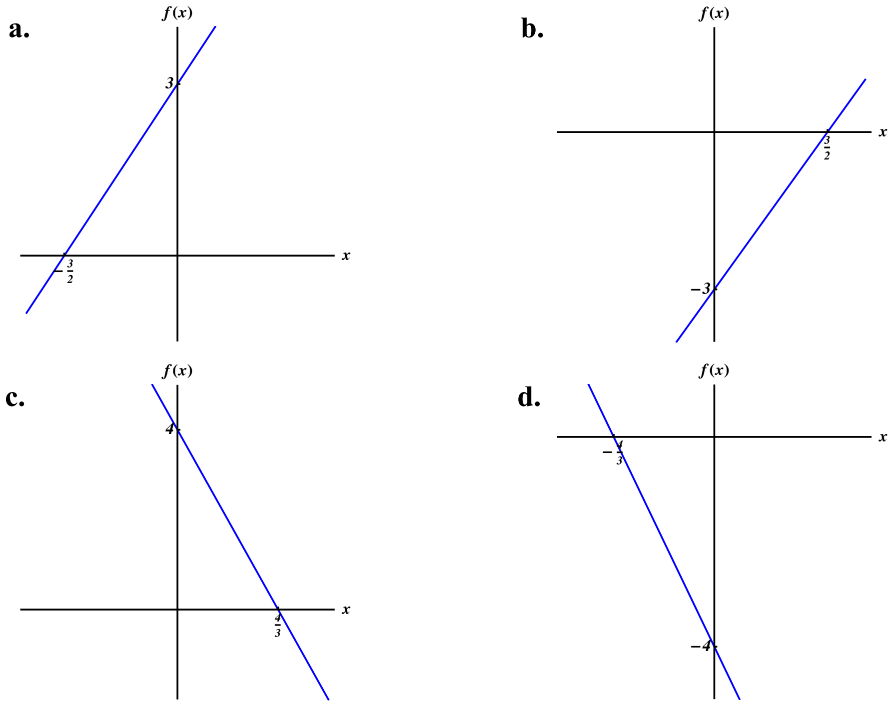


Figure 22.1: Graphs of functions in question 1

2. Calculate the x- and y-intercept for the following linear functions.

a. $f(x) = 4 + x$
answer: x-intercept: -4, y-intercept: 4

b. $f(x) = -5 + 3x$
answer: x-intercept: $5/3$, y-intercept:-5

c. $f(x) = 1 - 6x$
answer: x-intercept: $1/6$, y-intercept:1

d. $f(x) = -3 - 7x$
answer: x-intercept: $-3/7$, y-intercept:-3

3. Sketch the graphs of each pair of the following basic functions in one figure.

   a. $f(x) = -5 + 4x$, $g(x) = 2 + 4x$
   b. $f(x) = 4 + 2x$, $g(x) = 4 + x$

   answer:

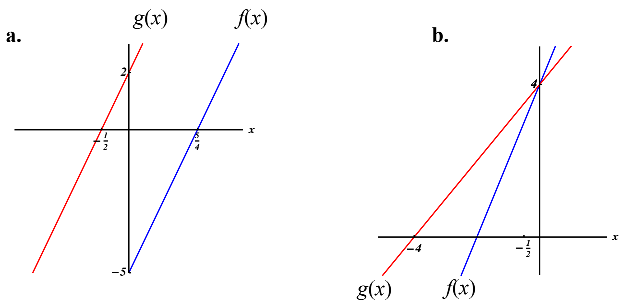


Figure 22.2: Graphs of functions in question 3

4. For which parameter combinations do the following functions have a positive x-intercept?

   a. $f(x) = \alpha \cdot x + 4$
   answer: $\alpha < 0$
   b. $f(x) = 2 \cdot x - \beta$
   answer: $\beta > 0$
   c. $f(x) = \rho \cdot x + \gamma$
   answer: There are two possibilities: (1) $\rho > 0$ and $\gamma < 0$, or (2) $\rho < 0$ and $\gamma > 0$

5. For which parameter combinations do the following functions have a positive y-intercept?

   a. $f(x) = \alpha \cdot x + 4$
   answer: The y-intercept is always positive, regardless the value of α.
   b. $f(x) = 2 \cdot x + \beta$
   answer: $\beta > 0$
   c. $f(x) = \gamma + \eta \cdot x$
   answer: $\gamma > 0$, η may have any value.

6. For which parameter combinations do the following pairs of functions have the same slope?

   a. $f(x) = \alpha \cdot x + 4$, $g(x) = 2x - 5 + x$
   answer: $\alpha = 3$
   b. $f(x) = \alpha \cdot x + 2$, $g(x) = \beta \cdot x - 4 + 3\alpha$
   answer: $\alpha = \beta$
   c. $f(x) = \alpha \cdot x - \beta \cdot x + 1$ $g(x) = 3x - 2$
   answer: $\alpha - \beta = 3$

section 17.1.2

1. Indicate whether the following functions have a maximum or a minimum, give the x-value of the extremum, and calculate their intercepts for both axes.
   a. $f(x) = x^2 - 2x + 3$
      answer:
      Extremum is a minimum, at $x = 1$. y-intercept at 3. No x-intercepts.
   b. $f(x) = 4 + 4x - 2x^2$
      answer:
      Extremum is a maximum, at $x = 1$. y-intercept at 4. x-intercepts at $1 \pm \sqrt{3}$.
   c. $f(x) = 6 - 2x + 9x^2$
      answer:
      Extremum is a minimum, at $x = 1/9$. y-intercept at 6. No x-intercepts.
2. Indicate whether the following functions have a maximum or a minimum, give the x-value of the extremum, and calculate their y-intercept.
   a. $f(x) = \alpha \cdot x^2 - 3x + 1$, with $\alpha > 0$
      answer:
      The extremum is a minimum, and occurs at $x = \frac{3}{2 \cdot \alpha}$. The y-intercept lies at 1.
   b. $f(x) = 2 + \beta \cdot x - 4x^2$
      answer:
      The extremum is a maximum, at $x = \frac{\beta}{8}$. The y-intercept lies at 2.
   c. $f(x) = 3x^2 - 2x + \gamma$
      answer:
      The extremum is a minimum, and occurs at $x = \frac{1}{3}$. The y-intercept lies at γ.
3. For which parameter combinations do the following functions have a maximum?
   a. $f(x) = 3x^2 + \alpha \cdot x - 1$
      answer: There is no α for which this function has a maximum.
   b. $f(x) = 3 - 2x - \beta \cdot x^2$
      answer: $\beta > 0$
   c. $f(x) = (1 - \gamma)x^2 - x + 4$
      answer: $1 - \gamma < 0 \Rightarrow \gamma > 1$
4. For which parameter combinations do the following functions have two x-intercepts?
   a. $f(x) = \alpha \cdot x^2 - 3x + 1$
      answer: The discriminant is $9 - 4\alpha$. This is larger than zero if $\alpha < 9/4$.
   b. $f(x) = x^2 + \gamma \cdot x - 3$
      answer: The discriminant is $\gamma^2 + 12$. This is always larger than zero, so there are two x-intercepts for all values of γ.
   c. $f(x) = 3x^2 - 4x + \beta$
      answer: The discriminant is $16 - 12\beta$. This is positive if $\beta < 4/3$

section 17.1.3

1. Determine the number of extrema of the following polynomials, using the derivative. Describe the general shape of each of the functions.
   a. $f(x) = x^3 - 3x^2 + 4x - 3$
      answer:
      The derivative is: $f'(x) = 3x^2 - 6x + 4$, there are no extrema. Since $a_3 = 1$ the function increases in x, and has an inflection point with a positive slope.

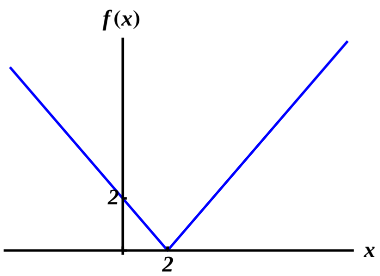
b. $f(x) = x^3 + 3x^2 - x - 4$
answer:
The derivative is: $f'(x) = 3x^2 + 6x - 1$, there are two extrema. Since $a_3 = 1$ the function initially increases in x, so its first extremum is a maximum, and its second a minimum. Its inflection point has a negative slope.

c. $f(x) = x^3 - 3x^2 + 3x - 2$
answer:
The derivative is: $f'(x) = 3x^2 - 6x + 3$, there are no extrema. Since $a_3 = 1$ the function increases in x, and has an inflection point with a positive slope.

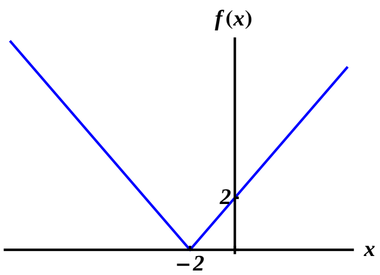
section 17.2

1. Sketch graphs of the following functions:

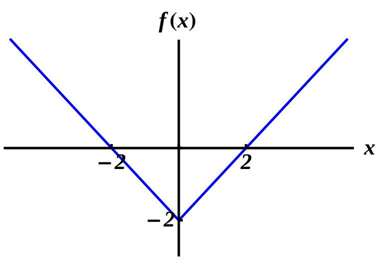
a. $f(x) = |x - 2|$
answer:



b. $f(x) = |x + 2|$
answer:



c. $f(x) = |x| - 2$
answer:



2. For which values of x are the following inequalities satisfied?

a. $|x| > 1$
answer: $x > 1$ or $x < -1$

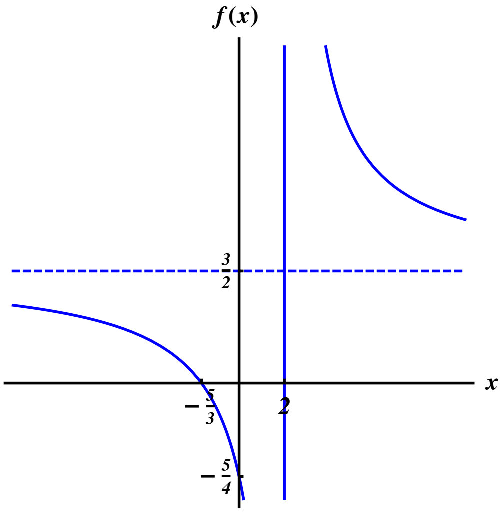
b. $|x+2| < 1$
answer: $x+2 > -1$ and $x+2 < 1$, so $x > -3$ and $x < -1$, i.e. $-3 < x < -1$

c. $|x-1| \geq 2$
answer: $x-1 \geq 2$ or $x-1 \leq -2$, so $x \geq 3$ or $x \leq -1$

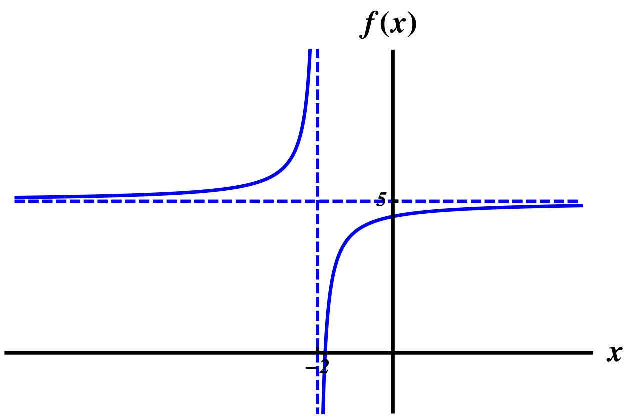
section 17.3

1. For each of the following functions, determine the horizontal and vertical asymptotes, determine its intercepts, and whether it is increasing or decreasing in x, and sketch its graph.

a. $f(x) = \frac{3x+5}{2x-4}$
answer:
Vertical asymptote: $x = 2$, Horizontal asymptote: $y = 3/2$. $ad = 3 \cdot (-4) = -12$, $cb = 2 \cdot 5 = 10$, so $ad < cb$. The function decreases in x. Its y-intercept is $y = -5/4$ and its x-intercept $x = -5/3$.



b. $f(x) = \frac{5x+9}{x+2}$
answer:
Vertical asymptote: $x = -2$, Horizontal asymptote: $y = 5$. $ad = 5 \cdot 2 = 10, cb = 1 \cdot 9 = 9$, so $ad > cb$.The function increases in x. Its y-intercept is $y = 9/2$ and its x-intercept $x = -9/5$.



2. Determine the vertical and horizontal asymptotes of each of the following functions.

a. $f(x) = \frac{\alpha \cdot x + 3}{1-2x}$, $\alpha \neq 0$
answer:
Vertical: $x = 1/2$, horizontal: $y = -\frac{\alpha}{2}$

b. $f(x) = \dfrac{x-4}{\beta \cdot x + 3}$, $\beta \neq 0$
answer:
Vertical: $x = -\frac{3}{\beta}$, horizontal: $y = \frac{1}{\beta}$

c. $f(x) = \dfrac{\gamma \cdot x - 5}{\beta \cdot x + 1}$, γ, $\beta \neq 0$
answer:
Vertical: $x = -\frac{1}{\beta}$, horizontal: $y = \frac{\gamma}{\beta}$

d. $f(x) = \dfrac{3x+2}{\alpha \cdot x - 2\beta}$, α, $\beta \neq 0$
answer:
Vertical: $x = \frac{2\beta}{\alpha}$, horizontal: $y = \frac{3}{\alpha}$

section 17.4

1. Simplify the following expressions as far as possible:

a. $(x^4)^2 \cdot x^5$
answer: x^{13}

b. $(x^{-3})^3$
answer: x^{-9}

c. $x^5 + x^3$
answer: Cannot be simplified.

d. $x^8 \cdot x^2$
answer: x^{10}

2. For each of the following functions indicate whether they are increasing or decreasing in x, and whether they are concave, or convex.

a. $f(x) = 2 \cdot x^{-3}$
answer: Decreases in x, convex.

b. $f(x) = -4 \cdot x^{1/2}$
answer: Decreases in x, convex.

c. $f(x) = x^{5/4}$
answer: Increases in x, convex.

d. $f(x) = x^{-3/2}$
answer: Decreases in x, convex.

e. $f(x) = -x^{7/3}$
answer: Decreases in x, concave.

3. For each of the following functions, indicate whether they are increasing or decreasing in x, and, when applicable give the horizontal and vertical asymptotes.

a. $f(x) = 1 + 3 \cdot x^{-4}$
answer: Decreasing, vertical asymptote at $x = 0$, horizontal asymptote at $y = 1$.

b. $f(x) = 2 - x^2$
answer: Decreasing, no asymptotes.

c. $f(x) = 3 - \dfrac{1}{x^{1/3}}$
answer:
Increasing, vertical asymptote at $x = 0$ horizontal asymptote at $y = 3$.

4. Simplify the following expressions as far as possible.

(a) $(x^a)^2 \cdot x^b$
answer: x^{2a+b}

(b) $\frac{x^2}{x^b}$
answer: x^{2-b}

(c) $x^{-3} \cdot x^a + x^a$
answer: $x^{a-3} + x^a$

5. For which parameter values are the following functions concave? In each case we only consider non-negative values of x.

(a) x^{a-2}
answer: $0 < a - 2 < 1$, so $2 < a < 3$

(b) x^{a-b}
answer: $0 < a - b < 1$, so $a > b$ and $a < 1 - b$

(c) $-x^{2a}$
answer: If $a > 1/2$ or $a < 0$

17.5

1. Simplify the following expressions as far as possible.

a. $e^{-3x} \cdot e^{3x}$
answer:
$= e^0 = 1$

b. $e^{-2x/3}$
answer:
Cannot be simplified.

c. $(e^{-\frac{1}{2}x})^{-1}$
answer:
$= e^{\frac{1}{2}x} (= \sqrt{e^x})$

2. For each of the following functions indicate whether they are increasing or decreasing in x, give their y-intercepts, and horizontal asymptote.

a. $f(x) = 1 + e^{2x}$
answer:
Increasing, $f(0) = 2$ horizontal asymptote at $y = 1$

b. $f(x) = 2 \cdot e^{-3x} - 2$
answer:
Decreasing, $f(0) = 0$ horizontal asymptote at $y = -2$

c. $f(x) = 1 - e^{-x}$
answer:
Increasing, $f(0) = 0$, horizontal asymptote at $y = 1$

3. For which parameter combinations are the following functions increasing in x?

a. $f(x) = 1 + b \cdot e^{-x}$
answer: $b < 0$

b. $f(x) = 3e^{a \cdot x} - 4$
answer: $a > 0$

c. $f(x) = e^{a \cdot x} \cdot e^{b \cdot x}$
answer: $a + b > 0$

section 17.6

1. Simplify the following expressions as far as possible.
   a. $\ln\left(e \cdot \sqrt{\frac{3}{2}}\right)$
   answer: $= \ln e + \frac{1}{2}\ln\left(\frac{3}{2}\right) = 1 + \frac{1}{2}\ln\left(\frac{3}{2}\right)$
   b. $2\ln(3) + \ln(4)$
   answer: $= \ln(36)$
   c. $\ln(\frac{1}{e})$
   answer: $= -1$
   d. $3\ln(e\sqrt{e})$
   answer: $= 3\ln\left(e^{3/2}\right) = \frac{9}{2}\ln(e) = \frac{9}{2}$
   e. $\ln(e \cdot 3x)$
   answer:
   $= 1 + \ln(3x)$
   f. $4 + \ln(3)$
   answer:
   Cannot be simplified.
2. Express the following equations with natural logarithms:
   a. $2^4 = 16$
   answer: $4\ln(2) = \ln(16)$
   b. $125 = 5^3$
   answer: $\ln(125) = 3\ln(5)$
   c. $9^{1/2} = 27$
   answer: $\frac{1}{2}\ln(9) = \ln(27)$
   d. $81 = \left(\frac{1}{3}\right)^{-4}$
   answer: $\ln(81) = -4\ln(1/3)(= 12\ln(3))$
   e. $c = a^5$, with $c > 0$, $a > 0$
   answer: $\ln(c) = 5\ln(a)$
   f. $p^q = 3r$, with $p > 0$, $r > 0$
   answer: $q\ln(p) = \ln(3r)(= \ln(3) + \ln(r))$
3. Rewrite the following expressions as single logarithms.
   a. $\ln(2) + \ln(3)$
   answer:
   $= \ln(6)$
   b. $\frac{1}{2}\ln(x) - \ln(y)$
   answer: $= \ln\left(\frac{\sqrt{x}}{y}\right)$
   c. $2\ln(a) - \ln(b)$
   answer:
   $= \ln\left(\frac{a^2}{b}\right)$
   d. $\ln(p) + 2\ln(q)$
   answer:
   $= \ln(p \cdot q^2)$
4. For each of the following functions determine the asymptote.
   a. $f(x) = \ln(1 - x)$
   answer:
   Asymptote at $x = 1$

b. $f(x) = 3\ln\left(x^2\right)$
   answer:
   Asymptote at $x = 0$
c. $f(x) = \ln(2 + x)$
   answer:
   Asymptote at $x = -2$

section 17.7

1. Rewrite the following functions as expressions with $\sin(x)$ and/or $\cos(x)$.
   a. $f(x) = \sin(-x)$
      answer: $f(x) = -\sin(x)$
   b. $f(x) = \cos(-x)$
      answer: $f(x) = \cos(x)$
   c. $f(x) = \sin(x + \frac{1}{2}\pi)$
      answer: $f(x) = \cos(x)$
   d. $f(x) = \cos(x + \frac{1}{2}\pi)$
      answer: $f(x) = -\sin(x)$
   e. $f(x) = \sin(x + \pi)$
      answer: $f(x) = -\sin(x)$
   f. $f(x) = \cos(x + \pi)$
      answer: $f(x) = -\cos(x)$
2. Give the amplitudes and period lengths of the following functions:
   a. $f(x) = 3\sin(x) + 1$
      answer: Amplitude: 3, period length: 2π.
   b. $f(x) = 2 - 2\cos(x)$
      answer: Amplitude: 2, period length: 2π.
   c. $f(x) = \sin(5x)$
      answer: Amplitude: 1, period length: $\frac{2}{5}\pi$.
   d. $f(x) = 2\cos(x/3)$
      answer: Amplitude: 2, period length: 6π.
   e. $f(x) = 3 + a \cdot \cos(x)$
      answer: Amplitude: a, period length: 2π.
   f. $f(x) = a + \sin(b \cdot x)$
      answer: Amplitude: 1, period length: $\frac{2\pi}{b}$.
3. Give the ranges of the following functions:
   a. $f(x) = 2 + \sin(x)$
      answer: $x \in [1, 3]$
   b. $f(x) = 1 - \cos(3 \cdot x)$
      answer: $x \in [0, 2]$
   c. $f(x) = 3\sin(x/4)$
      answer: $x \in [-3, 3]$
   d. $f(x) = a + \sin(2x)$
      answer: $x \in [a - 1, a + 1]$
   e. $f(x) = 1 + b \cdot \cos(x)$, with $b > 0$
      answer: $x \in [1 - b, 1 + b]$

f. $f(x) = a \cdot \cos(x/b) - 2$, with $a > 0$
answer: $x \in [-2 - a, -2 + a]$

section 17.8

1. Find the following limits.

a. $\lim_{x\to\infty} \frac{x^2 - 1}{2x - 3x^2}$
answer: $-\frac{1}{3}$

b. $\lim_{n\to\infty} \left(\frac{1}{3}\right)^n$
answer: 0

Chapter 18 Solving equations and inequalities

section 18.2

1. Solve the following equations for x, and write down all the steps with conditions where needed.

   a. $(1-a)\cdot x + 2x = 5$
   answer (example):

$$\begin{aligned} (1-a)\cdot x + 2x &= 5 \\ x\cdot(1-a+2) &= 5 \\ x\cdot(3-a) &= 5 \\ x &= \frac{5}{3-a} && \text{divide both sides by } 3-a \\ & && \text{condition: } a \neq 3 \end{aligned}$$

   b. $\alpha \cdot x - 3 = 2x + 4$
   answer (example):

$$\begin{aligned} \alpha\cdot x - 3 &= 2x+4 \\ \alpha\cdot x - 2x - 3 &= 4 && \text{subtract } 2x \text{ on both sides} \\ (\alpha-2)\,x &= 7 && \text{add 3 on both sides and rearrange} \\ x &= \frac{7}{\alpha-2} && \text{divide both sides by } \alpha-2 \\ & && \text{condition: } \alpha \neq 2 \end{aligned}$$

   c. $-\alpha x + 3 = \gamma x - 5$
   answer (example):

$$\begin{aligned} -\alpha\cdot x - \gamma x + 3 &= -5 && \text{subtract } \gamma x \text{ on both sides} \\ (-\alpha-\gamma)\,x &= -8 && \text{subtract 3 on both sides and rearrange} \\ x &= \frac{-8}{-\alpha-\gamma} && \text{divide both sides by } -\alpha-\gamma \\ x &= \frac{8}{\alpha+\gamma} \\ & && \text{condition: } \alpha \neq -\gamma \end{aligned}$$

2. Solve the following equations for x, and write down all the steps with conditions where needed.

   a. $a\cdot x - x + b = c$

answer (example):

$$\begin{aligned} a \cdot x - x + b &= c \\ (a-1) \cdot x + b &= c \\ (a-1) \cdot x &= c - b && \text{subtract } b \text{ on both sides} \\ x &= \frac{c-b}{a-1} && \text{divide by } (a-1) \text{ on both sides,} \\ & && \text{condition: } a \neq 1 \end{aligned}$$

b. $c \cdot x - d = a \cdot x - b$
answer (example):

$$\begin{aligned} c \cdot x - d &= a \cdot x - b \\ c \cdot x - a \cdot x - d &= -b && \text{subtract } a \cdot x \text{ on both sides} \\ c \cdot x - a \cdot x &= d - b && \text{add } d \text{ on both sides} \\ (c-a) \cdot x &= d - b \\ x &= \frac{d-b}{c-a} && \text{divide by } c-a \text{ on both sides} \\ & && \text{condition: } c \neq a \end{aligned}$$

3. Solve the following equations for ρ, and write down all the steps with conditions where needed.

a. $a \cdot \rho - 2b = 5\rho$
answer:

$$\begin{aligned} a \cdot \rho - 2b &= 5\rho \\ a \cdot \rho - 5\rho - 2b &= 0 && \text{subtract } 5\rho \text{ from both sides} \\ \rho\,(a-5) &= 2b && \text{add } 2b \text{ on both sides} \\ \rho &= \frac{2b}{a-5} && \text{divide both sides by } a-5 \\ & && \text{condition: } a-5 \neq 0 \end{aligned}$$

b. $x \cdot \rho - 3 = 2x + 1$
answer:

$$\begin{aligned} x \cdot \rho - 3 &= 2x + 1 \\ x \cdot \rho &= 2x + 4 && \text{add 3 on both sides} \\ \rho &= \frac{2x+4}{x} && \text{divide both sides by } x \\ & && \text{condition: } x \neq 0 \end{aligned}$$

section: 18.3

1. Solve the following equations for x:

a. $2x^2 - 3x = 1$
answer: $x = \frac{1}{4}\left(3 \pm \sqrt{17}\right)$

b. $x^2 = x + 2$
answer: $x = -1$, or $x = 2$

c. $3x^2 + 2x = x + 2$
answer: $x = -1$, or, $x = \frac{2}{3}$

2. Solve the following equation for x. Add conditions on the parameters where necessary: $ax^2 + x = 2b$
answer:

$$\begin{aligned} ax^2 + x &= 2b \\ ax^2 + x - 2b &= 0 && \text{subtract } 2b \text{ on both sides} \\ x &= \frac{-1 \pm \sqrt{1+8ab}}{2a} && \text{use abc - formula} \\ \text{conditions: } a \neq 0,\ 8ab \geq -1 &\Rightarrow ab \geq -\frac{1}{8} \end{aligned}$$

If $a = 0$ the solution is $x = 2b$.

3. Solve the following equation for x, and list all possible outcomes for different values of p:
$p \cdot x^2 + 2x - 1 = 0, p \neq 0$
answer: The discriminant is: $\sqrt{4+4p}$ The equation has no real roots if $p < -1$. If $p = -1$ there is a single root, at $x = -1$. If $p > -1$ there are two real roots:

$$x = \frac{-2 \pm \sqrt{4+4p}}{2p} = \frac{-1 \pm \sqrt{1+p}}{p}$$

4. Solve the following equations for α.

   a. $2\alpha^2 - 4\alpha + 2 = 0$
   answer: The discriminant is 0. There is a single root at $\alpha = 1$.

   b. $x^2 \cdot \alpha - 2x + 3 = 0$
   answer: Note: since α is the unknown, this is a *linear* equation. The solution is:

$$\alpha = \frac{2x-3}{x^2},\ x \neq 0$$

   If $x = 0$ there is no solution.

section 18.4

1. Solve the following equations for x. Write down all steps and add conditions where necessary.

   a. $\frac{1}{x} - 3 \cdot x = 0$
   answer:

$$\begin{aligned} \frac{1}{x} - 3 \cdot x &= 0, && \text{condition: } x \neq 0 \\ 1 - 3x^2 &= 0 && \text{multiply both sides with } x \\ -3x^2 &= -1 && \text{subtract 1 on both sides} \\ x^2 &= \frac{-1}{-3} = \frac{1}{3} && \text{divide by -3 on both sides} \\ x &= \pm\sqrt{\frac{1}{3}} \end{aligned}$$

   b. $\frac{1}{x+2} - 3 = 0$

answer:

$$\begin{aligned}
\frac{1}{x+2} - 3 &= 0, && \text{condition: } x \neq -2\\
1 - 3\,(x+2) &= 0 && \text{multiply both sides with } x+2\\
1 - 3x - 6 &= 0 && \text{work out brackets}\\
-3x - 5 &= 0 && \text{rearrange}\\
3x &= -5 && \text{add } 3x \text{ on both sides}\\
x &= -\frac{5}{3} && \text{divide by 3 on both sides}
\end{aligned}$$

c. $\frac{1}{x-1} = -3x$
answer:

$$\begin{aligned}
\frac{1}{x-1} &= -3x && \text{condition: } x \neq 1\\
1 &= -3x\,(x-1) && \text{multiply both sides by } x-1\\
1 &= -3x^2 + 3x && \text{work out brackets}\\
-3x^2 + 3x - 1 &= 0 && \text{subtract } -1 \text{ on both sides}\\
&\text{discriminant: } 9 - 12 < 0
\end{aligned}$$

so there are no solutions in $\mathbb{R}$.

d. $\frac{x+2}{x-1} = 3$
answer:

$$\begin{aligned}
\frac{x+2}{x-1} &= 3 && \text{condition: } x \neq 1\\
x + 2 &= 3\,(x-1) && \text{multiply both sides with } x-1\\
x + 2 &= 3x - 3 && \text{work out brackets}\\
5 &= 2x && \text{subtract } x \text{ and add 3 on both sides}\\
x &= \frac{5}{2}
\end{aligned}$$

2. Solve the following equations for x. Write down all steps and add conditions where necessary.

a. $\frac{1}{x+a} = -3$
answer:

$$\begin{aligned}
\frac{1}{x+a} &= -3, && \text{condition: } x \neq -a\\
1 &= -3\,(x+a) && \text{multiply both sides by } x+a\\
1 &= -3x - 3a && \text{work out brackets}\\
1 + 3a &= -3x && \text{add } 3a \text{ on both sides}\\
x &= -\frac{1+3a}{3} = -\frac{1}{3} - a && \text{divide by } -3 \text{ on both sides and rearrange}
\end{aligned}$$

Note that the condition is satisfied.

b. $\frac{1}{a \cdot x} = \frac{b}{x} + c$

answer:

$$\begin{aligned} \frac{1}{a \cdot x} &= \frac{b}{x} + c && \text{conditions: } a \neq 0, x \neq 0 \\ 1 &= b \cdot a + c \cdot a \cdot x && \text{multiply both sides by } a \cdot x \\ c \cdot a \cdot x &= 1 - b \cdot a && \text{subtract } b \cdot a \text{ from both sides} \\ x &= \frac{1 - b \cdot a}{c \cdot a} && \text{divide both sides by } c \cdot a \\ \text{conditions: } c \neq 0 \end{aligned}$$

Note that, furthermore there is a condition that $x \neq 0$, so the outcome must not be equal to zero, leading to the condition: $b \cdot a \neq 1$

c. $\frac{x}{x+a} - 3x = 0$
answer:

$$\begin{aligned} \frac{x}{x+a} - 3x &= 0 && \text{condition: } x \neq -a \\ x - 3(x+a) &= 0 && \text{multiply both sides by } x + a \\ -2x - 3a &= 0 \\ -3a &= 2x && \text{add } 2x \text{ on both sides} \\ x &= \frac{-3a}{2} && \text{divide both sides by 2} \end{aligned}$$

3. Solve the following equations for x. Write down the steps and add conditions where necessary.

a. $\frac{1}{x+a} - 2x = 0$
answer:

$$\begin{aligned} \frac{1}{x+a} - 2x &= 0 \\ 1 - 2x(x+a) &= 0 && \text{multiply both sides by } x + a \\ & && \text{condition: } x \neq -a \\ -2x^2 - 2a \cdot x + 1 &= 0 && \text{work out brackets, and rearrange} \end{aligned}$$

This is a quadratic equation. The discriminant is: $4a^2 + 8$, which is always positive, so there are two roots:

$$\begin{aligned} x &= \frac{2a \pm \sqrt{4a^2 + 8}}{-4} \\ &= \frac{-a \pm \sqrt{a^2 + 2}}{2} \end{aligned}$$

Since we have the condition that $x \neq -a$ outcome should not be equal to $-a$. This should be checked. For the largest root we have:

$$\begin{aligned} \frac{-a + \sqrt{a^2 + 2}}{2} &\neq -a \\ -a + \sqrt{a^2 + 2} &\neq -2a && \text{multiply both sides by 2} \\ \sqrt{a^2 + 2} &\neq -a && \text{add } a \text{ on both sides} \end{aligned}$$

This is true for all a. For the other root:

$$\begin{aligned} \frac{-a-\sqrt{a^2+2}}{2} &\neq -a \\ -a-\sqrt{a^2+2} &\neq -2a && \text{multiply both sides by 2} \\ -\sqrt{a^2+2} &\neq -a && \text{add } a \text{ on both sides} \\ \sqrt{a^2+2} &\neq a && \text{multiply both sides by } -1 \end{aligned}$$

This is also true for all a.

b. $\frac{1}{x-a}+\frac{b}{x}=0$
answer:

$$\begin{aligned} \frac{1}{x-a}+\frac{b}{x} &= 0 \\ \frac{x}{x-a}+b &= 0 && \text{multiply both sides by } x \\ & && \text{condition: } x \neq 0 \\ x+b\,(x-a) &= 0 && \text{multiply both sides by } x-a \\ & && \text{condition: } x \neq a \\ x\,(1+b)-b\cdot a &= 0 && \text{rearrange} \\ x\,(1+b) &= b\cdot a && \text{add } b\cdot a \text{ on both sides} \\ x &= \frac{b\cdot a}{1+b} && \text{divide both sides by } 1+b \\ & && \text{condition: } b \neq -1 \end{aligned}$$

Since $x \neq 0$ the outcome must satisfy the condition:

$$\frac{b\cdot a}{1+b} \neq 0 \Rightarrow b \neq 0 \text{ and } a \neq 0$$

Another conditions was $x \neq a$, but this is satisfied since

$$\frac{b}{1+b}$$

is never equal to one, regardless the value of b.

section 18.5

1. Solve the following equations. Write down all the steps, and the conditions where needed.

a. $\sqrt{x-2}=3$
answer:

$$\begin{aligned} \sqrt{x-2} &= 3 && \text{condition: } x \geq 2 \\ x-2 &= 9 && \text{raise both sides to power 2} \\ x &= 11 && \text{add 2 on both sides} \end{aligned}$$

b. $\sqrt{x+1}-2\sqrt{x-1}=0$

answer:

$$\begin{aligned}
\sqrt{x+1} - 2\sqrt{x-1} &= 0 && \text{condition: } x \geq 1 \\
\sqrt{x+1} &= 2\sqrt{x-1} && \text{add } 2\sqrt{x-1} \text{ on both sides} \\
x+1 &= 4\,(x-1) && \text{raise both sides to power 2} \\
x+1 &= 4x-4 && \text{work out brackets} \\
5 &= 3x && \text{add 4 and subtract } x \text{ on both sides} \\
x &= \frac{5}{3} && \text{divide by 3 on both sides}
\end{aligned}$$

check conditions:

$$\frac{5}{3} > 1 \qquad \text{so the solution is valid}$$

2. Solve the following equations. Write down the steps and add conditions where necessary.
   a. $\sqrt{x} = 2 - x$
   answer:

$$\begin{aligned}
\sqrt{x} &= 2-x && \text{conditions: } x \geq 0,\ 2-x \geq 0 \Rightarrow x \leq 2 \\
x &= (2-x)^2 && \text{raise both sides to power 2} \\
x &= 4-4x+x^2 && \text{work out brackets} \\
4-5x+x^2 &= 0 && \text{subtract } x \text{ from both sides} \\
x &= \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm 3}{2} \\
x &= 4, \text{ or } x = 1
\end{aligned}$$

condition: $0 \leq x \leq 2$, so $x = 1$ is a valid solution

   b. $\sqrt{x} - 3 = 3\sqrt{x} - 2$
   answer:

$$\begin{aligned}
\sqrt{x} - 3 &= 3\sqrt{x} - 2 && \text{condition: } x \geq 0 \\
2\sqrt{x} - 2 &= -3 && \text{subtract } \sqrt{x} \text{ from both sides} \\
2\sqrt{x} &= -1 && \text{add 2 on both sides}
\end{aligned}$$

no solution

3. Solve the following equations for x. Write down the steps and add conditions where necessary.
   a. $\sqrt{a \cdot x - 1} - 3 = 0$
   answer:

$$\begin{aligned}
\sqrt{a \cdot x - 1} - 3 &= 0 && \text{condition: } a \cdot x \geq 1 \\
\sqrt{a \cdot x - 1} &= 3 && \text{add 3 on both sides} \\
a \cdot x - 1 &= 9 && \text{raise both sides to the power 2} \\
a \cdot x &= 10 && \text{add 1 on both sides} \\
x &= \frac{10}{a} && \text{divide both sides by } a \\
& && \text{condition: } a \neq 0
\end{aligned}$$

Check the condition for the outcome:

$$a \cdot \frac{10}{a} \geq 1 \qquad (22.1)$$

This is always satisfied, so the only condition is that a is not equal to zero.

b. $\sqrt{x-a} = 3b$
answer:

$$\begin{aligned} \sqrt{x-a} &= 3b && \text{conditions: } x \geq a \text{ and } b \geq 0 \\ x - a &= 9b^2 && \text{raise both sides to the power 2} \\ x &= 9b^2 + a && \text{add } a \text{ on both sides} \end{aligned}$$

Check the condition for the outcome:

$$9b^2 + a \geq a$$

This is always true.

section (18.6)

1. Derive the result in (18.29)
answer:

$$\begin{aligned} ax + b &= \ln(c), && c > 0 \\ ax &= \ln(c) - b && \text{subtract } b \text{ from both sides} \\ x &= \frac{\ln(c) - b}{a} && \text{divide both sides by } a, \text{ condition: } a \neq 0 \end{aligned}$$

2. Solve (18.31) for x.
answer:

$$\begin{aligned} 2x &= \ln(a) + 3x - 1, && a > 0 \\ -x &= \ln(a) - 1 && \text{subtract } 3x \text{ from both sides} \\ x &= 1 - \ln(a) && \text{multiply both sides by } -1 \end{aligned}$$

3. Solve for x: $ae^{bx} = \frac{1}{2}, a > 0$
answer:

$$\begin{aligned} ae^{bx} &= \frac{1}{2}, a > 0 \\ e^{bx} &= \frac{1}{2a} \\ bx &= -\ln(2a) \\ x &= \frac{-\ln(2a)}{b} \end{aligned}$$

section 18.7

1. Solve the following equations for x:
   a. $\cos(2x) = 1/2$
   answer: $2x = \frac{\pi}{3} + k \cdot 2\pi \Rightarrow x = \frac{\pi}{6} + k \cdot \pi$ or $2x = -\frac{\pi}{3} + k \cdot 2\pi \Rightarrow x = -\frac{\pi}{6} + k \cdot \pi$
   b. $\sin(x+3) = -1$
   answer: $x + 3 = \frac{3\pi}{2} + k \cdot 2\pi \Rightarrow x = \frac{3\pi}{2} - 3 + k \cdot 2\pi$
   c. $\tan(x/3) = 1$
   answer: $x/3 = \frac{\pi}{4} + k\pi \Rightarrow x = \frac{3\pi}{4} + 3k \cdot \pi$
   d. $2 \cdot \cos(x) = 1$
   answer: $\cos(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{3} + k \cdot 2\pi$ or $x = -\frac{\pi}{3} + k \cdot 2\pi$
2. Solve the following equations for x:

a. $\cos(a \cdot x) = 1, a \neq 0$
answer: $a \cdot x = k \cdot 2\pi \Rightarrow x = k \cdot \frac{2\pi}{a}$

b. $\sin(a \cdot x + b) = -1, a \neq 0$
answer: $a \cdot x + b = \frac{3\pi}{2} + k \cdot 2\pi \Rightarrow x = \frac{3\pi}{2a} - b + k \cdot \frac{2\pi}{a}$

section 18.8.1

1. Solve the following inequalities for x. Find all solutions:

a. $x + b > 2x - 3$
answer:

$$\begin{aligned} &x + b > 2x - 3 \\ &x + b + 3 > 2x \\ &b + 3 > x \end{aligned}$$

b. $c \cdot x + 1 \leq 2x + 3$
answer:

$$\begin{aligned} &c \cdot x + 1 \leq 2x + 3 \\ &c \cdot x - 2x \leq 2 \\ &(c - 2)\, x \leq 2 \\ &c > 2 : x \leq \frac{2}{c-2} \\ &c < 2 : x \geq \frac{2}{c-2} \\ &c = 2 : x \in \mathbb{R} \end{aligned}$$

c. $d \cdot x - 2 \geq 3x + 1$
answer:

$$\begin{aligned} &d \cdot x - 2 \geq 3x + 1 \\ &d \cdot x \geq 3x + 3 \\ &d \cdot x - 3x \geq 3 \\ &(d - 3)\, x \geq 3 \\ &d > 3 : x \geq \frac{3}{d-3} \\ &d < 3 : x \leq \frac{3}{d-3} \\ &d = 3 : \text{ no solution} \end{aligned}$$

2. Solve the following inequalities for x. Find all solutions:

a. $a \cdot x + 1 > x - b$

answer:

$$
\begin{aligned}
& a \cdot x + 1 > x - b \\
& a \cdot x > x - b - 1 \\
& a \cdot x - x > -b - 1 \\
& (a-1)\, x > -(b+1) \\
& a > 1 : x > \frac{-(b+1)}{a-1} \left(\Rightarrow x > \frac{b+1}{1-a} \right) \\
& a < 1 : x < \frac{-(b+1)}{a-1} \left(\Rightarrow x < \frac{b+1}{1-a} \right) \\
& a = 1 : \\
& \text{if } -(b+1) < 0 \text{ (so if } b > -1) : x \in \mathbb{R} \\
& \text{if } -(b+1) > 0 \text{ (so if } b < -1) : \text{ no solution} \\
& \text{if } b = -1 : \text{no solution}
\end{aligned}
$$

b. $a \cdot x + 2 \leq -c \cdot x + 3$

answer:

$$
\begin{aligned}
& a \cdot x + 2 \leq -c \cdot x + 3 \\
& a \cdot x + c \cdot x \leq 1 \\
& (a+c)\, x \leq 1 \\
& a + c > 0 : x \leq \frac{1}{a+c} \\
& a + c < 0 : x \geq \frac{1}{a+c} \\
& a + c = 0 : x \in \mathbb{R}
\end{aligned}
$$

section 18.8.2

1. Solve the following inequalities for x. Examine solutions for all possible parameter values:

a. $a \cdot x^2 > 2$

answer:

$$
\begin{aligned}
& a \cdot x^2 > 2 \\
& a > 0 : x^2 > \frac{2}{a} \Rightarrow x < -\sqrt{\frac{2}{a}} \text{ or } x > \sqrt{\frac{2}{a}} \\
& a \leq 0 : \text{no solution in } \mathbb{R}
\end{aligned}
$$

b. $a \cdot x^2 + 3 > 2$

answer:

$$
\begin{aligned}
& a \cdot x^2 + 3 > 2 \\
& a \cdot x^2 > -1 \\
& a \geq 0 : x \in R \\
& a < 0 : x^2 < -\frac{1}{a} \Rightarrow -\sqrt{-\frac{1}{a}} < x < \sqrt{-\frac{1}{a}}
\end{aligned}
$$

Note that in this case $-\frac{1}{a} > 0$, so its square root exists.

2. Solve the following inequality for x. Examine solutions for all possible parameter values: $a \cdot x^2 + 5x - 2 > 0, a \neq 0$

answer:

First examine roots of $a \cdot x^2 + 5x - 2 = 0$. The discriminant is: $25 + 8a$. So we have the following situations:

$$8a > -25: \text{ two roots: } x = \frac{-5 \pm \sqrt{25+8a}}{2a}$$
$$8a = -25 : \text{one root } x = -\frac{5}{2a}$$
$$8a < -25 \text{ no (real) roots}$$

In the first situation, we need to distinguish two situations:
If $\frac{-25}{8} < a < 0$ the parabola has an opening to the bottom, so the inequality holds for all x -values between the two roots:

$$\frac{-5-\sqrt{25+8a}}{2a} < x < \frac{-5+\sqrt{25+8a}}{2a}$$

If $a > 0$ the parabola has an opening to the top, so the inequality holds for $x < \frac{-5-\sqrt{25+8a}}{2a}$ or $x > \frac{-5+\sqrt{25+8a}}{2a}$.
If $a = \frac{-25}{8}$ the parabola has an opening to the bottom, and its maximum value is zero, so there is no x-value for which the inequality holds.
If $a < \frac{-25}{8}$ the parabola has an opening to the bottom, and lies entirely below the horizontal axis, so there is no x-value for which the inequality holds.

section 18.9

1. Solve the following systems of equations for x and y:

   a.
   $$3x = 2y + x + 1$$
   $$x + y = 4$$

   answer: $x = 9/4, y = 7/4$

   b.
   $$x - y = -1$$
   $$6x - 7y = 3$$

   answer: $x = -10, y = -9$

   c.
   $$x - y = x + 3y$$
   $$-2y = 7 + x$$

   answer: $x = -7, y = 0$

2. Solve the following systems of equations for x and y:

   a.
   $$x - y = 2$$
   $$2x - 2y = 5$$

   answer: There is no solution.

   b.
   $$3x - 9y = 6$$
   $$x = 3y + 2$$

   answer: There is an infinite number of solutions, namely all x, y combinations such that $x = 3y + 2$.

section 18.9.2

1. Find all solutions of the following system of equations.

$$x \cdot (x-1) - 3x \cdot y = 0$$
$$x + y = 1$$

answer: The first equation gives two possibilities: $x = 0$ or $x - 1 - 3y = 0 \Rightarrow x = 1 + 3y$. Substitution of $x = 0$ in the second equation gives $y = 1$,so one solution is $x = 0, y = 1$. Substitution of $x = 1 + 3y$ in the second equation gives $1 + 4y = 1 \Rightarrow y = 0$, so the other solution is $x = 1, y = 0$.

2. Find all solutions of the following system of equations.

$$\frac{x \cdot y}{x+2} - 5y = y$$
$$y - \frac{x \cdot y}{x+2} = 0$$

answer: For both equaitons the condition $x \neq -2$ holds. Rearranging the first equation gives:

$$\frac{x \cdot y}{x+2} - 6y = 0 \Rightarrow y\left(\frac{x}{x+2} - 6\right) = 0$$

So there are two possibilities: $y = 0$ or $\frac{x}{x+2} = 6$. Rearranging of the second equation gives:

$$y\left(1 - \frac{x}{x+2}\right) = 0$$

This equation is satisfied when $y = 0$ or $\frac{x}{x+2} = 1$. So both equations are satisfied when $y = 0$. In that case x may have any value except -2. Note that there are no x-values for which both $\frac{x}{x+2} = 6$ and $\frac{x}{x+2} = 1$ holds, so there are no solutions with other y-values possible.

Chapter 19 Differentiation

section 19.1

1. Compute the derivatives of the following functions:

   a. $f(x) = e^{x^2}$
   answer: Using the chain rule: $f'(x) = 2x \cdot e^{x^2}$.

   b. $f(x) = (x-2)(2x+3)$
   answer: Using the product rule: $f'(x) = 2x + 3 + 2 \cdot (x-2) = 2x + 3 + 2x - 4 = 4x - 1$.

   c. $f(x) = \sin(3x)$
   answer: Using the chain rule: $f'(x) = 3\cos(3x)$.

   d. $f(x) = 3x\cos(5x)$
   answer: Using the product rule and the chain rule: $f'(x) = 3\cos(5x) + 3x \cdot (5 \cdot (-\sin(5x))) = 3\cos(5x) - 15x\sin(5x)$.

   e. $f(x) = \dfrac{x+1}{x^2-2}$
   answer: Using the quotient rule: $f'(x) = \dfrac{(x^2-2) - 2x \cdot (x+1)}{(x^2-2)^2} = \dfrac{-x^2-2x-2}{(x^2-2)^2}$

   f. $f(x) = e^x(2x^3 - x^2)$
   answer: Using the product rule: $f'(x) = e^x(2x^3 - x^2) + e^x(6x^2 - 2x)$

2. Compute the derivatives of the following functions:

   a. $f(x) = \alpha \cdot x^2 + \beta \cdot x$
   answer: $f'(x) = 2\alpha \cdot x + \beta$

   b. $f(x) = 3e^{a \cdot x}$
   answer: Using the chain rule: $f'(x) = 3a \cdot e^{a \cdot x}$

   c. $f(x) = e^{a \cdot x + b}$
   answer: Using the chain rule: $f'(x) = a \cdot e^{a \cdot x + b}$

   d. $f(x) = a \cdot x \cdot e^{b \cdot x}$
   answer: Using the product rule and the chain rule: $f'(x) = a \cdot e^{b \cdot x} + a \cdot b \cdot x \cdot e^{b \cdot x}$

section 19.2

1. Match the graph of each function in (a)-(d) of Fig. 19.5 with the graph of its derivative in I-IV. Motivate your choices.   Match intervals where the function decreases with those where the derivative is negative, and intervals where the function increases with those where the derivative is positive. Points where the derivative is zero correspond to extrema or inflection points: (a) II, (b) IV , (c) I ,(d) III.

2. Compute the first and second derivatives of the following functions:

   a. $f(x) = 4x - 3x^2$

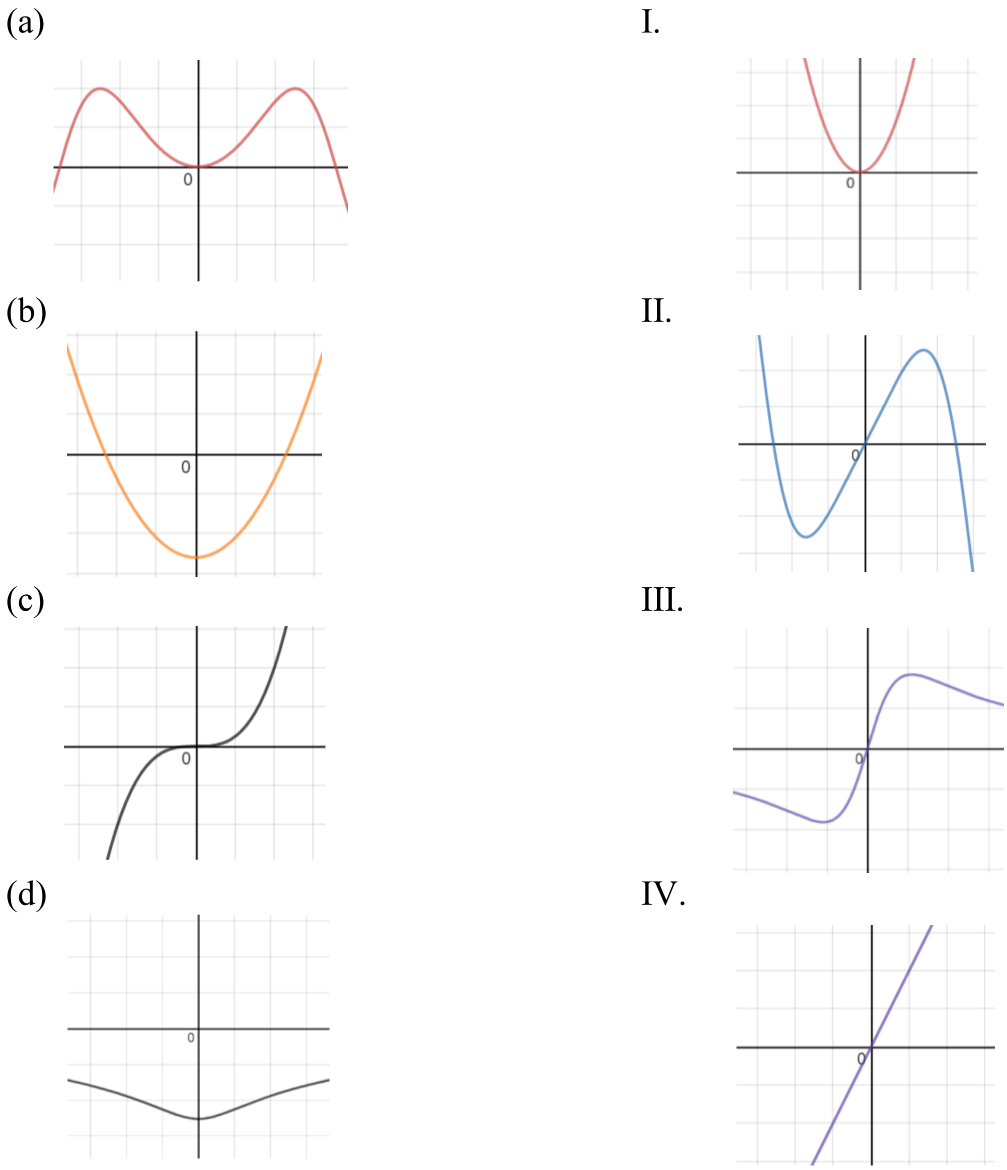


Figure 22.3: Exercise 1

answer:

$$f'(x) = 4 - 6x$$
$$f''(x) = -6$$

b. $f(x) = \sqrt{x}$

answer:

$$f'(x) = \frac{1}{2\sqrt{x}}$$
$$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}$$

c. $f(x) = \frac{3}{x}$

answer:

$$f'(x) = -\frac{3}{x^2}$$
$$f''(x) = \frac{6}{x^3}$$

d. $f(x) = 3^x$

answer:

$$f'(x) = 3^x \ln(3)$$
$$f''(x) = 3^x \ln^2(3)$$

e. $f(x) = 6x^2 - 3x + 2$

answer:

$$f'(x) = 12x - 3$$
$$f''(x) = 12$$

f. $f(x) = x^{-2}$

answer:

$$f'(x) = -2x^{-3}$$
$$f''(x) = 6x^{-4}$$

g. $f(x) = x^{\frac{3}{2}}$

answer:

$$f'(x) = \frac{3\sqrt{x}}{2}$$
$$f''(x) = \frac{3}{4\sqrt{x}}$$

section 19.3

1. Compute the partial derivatives to x and to y for the following functions:

a. $f(x,y) = 2x \cdot y + 3x - y$

answer:

$$\frac{\partial f(x,y)}{\partial x} = 2y + 3$$
$$\frac{\partial f(x,y)}{\partial y} = 2x - 1$$

b. $f(x,y) = a \cdot x^2 + b \cdot y^2$

answer:

$$\frac{\partial f(x,y)}{\partial x} = 2a \cdot x$$
$$\frac{\partial f(x,y)}{\partial y} = 2b \cdot y$$

c. $f(x,y) = x \cdot e^y$
answer:

$$\frac{\partial f(x,y)}{\partial x} = e^y$$
$$\frac{\partial f(x,y)}{\partial y} = x \cdot e^y$$

section 19.4.1

1. What is the second order approximation of the function $f(x)$ in (19.19) in the point $x = -1$?
answer: Since $f''(-1) = 0$ the second order term vanishes, and the approximation is: $f(x) \approx \ln(2) - (x+1)$

2. What is the general expression for the second order approximation of the function $f(x)$ in (19.19) in the point $x = a$?
answer: $f(x) \approx \ln\left(a^2+1\right) + \frac{2a}{a^2+1} \cdot (x-a) + \frac{1}{2} \cdot \left(\frac{-2a^2+2}{(a^2+1)^2}\right) \cdot (x-a)^2$

3. Give the first order approximation of the function: $f(x) = e^{x^2}$ in the point $x = 1$
answer: $f(x) \approx e + 2e \cdot (x-1)$

4. Give the second order approximation of the function: $f(x) = e^{x^2}$ in the point $x = 1$
answer: $f(x) \approx e + 2e \cdot (x-1) + \frac{1}{2} \cdot (2e+4e) \cdot (x-1)^2$
$= e + 2e \cdot (x-1) + 3e \cdot (x-1)^2$

Chapter 20 Integration

section 20.1

1. Give the outcomes of the following integrals:

   a. $2x^3 \mathrm{d}x$

   answer:
   $\frac{1}{2}x^4 + c$

   b. $\int \left(x^2 + 4x - 1\right) \mathrm{d}x$

   answer:
   $\frac{1}{3}x^3 + 2x^2 - x + c$

   c. $\int \left(5x^2 + 6\right) \mathrm{d}x$

   answer:
   $\frac{5}{3}x^3 + 6x + c$

2. Give the outcomes of the following integrals:

   a. $\int 4e^{3x} \mathrm{d}x$

   answer:
   $\frac{4}{3}e^{3x} + c$

   b. $\int \frac{2}{x-1} \mathrm{d}x$

   answer:
   $2 \ln |x - 1| + c$

   c. $\int \frac{1}{2x+3} \mathrm{d}x$

   answer:
   $\frac{1}{2} \ln |2x + 3| + c$

section 20.2

1. Give the outcomes of the following integrals:

   a. $\int\limits_2^3 (2x + 5) \mathrm{d}x$

   answer:
   $\left[x^2 + 5x\right]_2^3 = 9 + 15 - (4 + 10) = 10$

   b. $\int\limits_1^5 \frac{1}{2x} \mathrm{d}x$

   answer:
   $\left[\frac{1}{2} \ln |x|\right]_1^5 = \frac{1}{2} (\ln (5) - 0) = \frac{1}{2} \ln (5)$

   c. $\int\limits_0^4 2e^{5x} \mathrm{d}x$

   answer:
   $\left[\frac{2}{5}e^{5x}\right]_0^4 = \frac{2}{5} \left(e^{20} - e^0\right) \approx 1.94 \cdot 10^8$

section 20.3.2

1. Compute the following integrals, using substitution:

a. $\int \frac{\ln(t)}{t} dt$

answer: Since $\ln(t)$ is the antiderivative of $1/t$, we can use substitution: Define $y = \ln(t)$, then $dy = (1/t)dt$. So the integral becomes:

$$\int \frac{\ln(t)}{t} dt =$$
$$\int y dy = \frac{1}{2} y^2$$

Substitution of $y = \ln(t)$ gives the outcome:

$$\int \frac{\ln(t)}{t} dt = \frac{1}{2} (\ln(t))^2 + c$$

b. $\int_0^\pi \sin(x) \cdot \cos(x)\, dx$

answer: Since $\cos(x)$ is the derivative of $\sin(x)$, substitution of $y = \sin(x)$ gives:

$$\int \sin(x) \cdot \cos(x)\, dx = \int y dy = \frac{1}{2} y^2$$

and thus

$$\int_0^\pi \sin(x) \cdot \cos(x)\, dx = \left[\frac{1}{2} (\sin(x))^2\right]_0^\pi = 0$$

c. $\int_0^1 x \cdot e^{x^2} dx$

answer: The derivative of x^2 is $2x$, and $x dx = \frac{1}{2} dy$, so substitution of $y = x^2$ gives:

$$\int x \cdot e^{x^2} dx = \frac{1}{2} \int e^y dy = \frac{1}{2} e^y$$

and thus

$$\int_0^1 x \cdot e^{x^2} dx = \left[\frac{1}{2} e^{x^2}\right]_0^1 = \frac{1}{2}(e - 1)$$

section 20.3.3

1. Compute the following integrals, using partial integration:

a. $\int x \cdot e^x dx$

answer: Let $f(x) = x$ and$g'(x) = e^x$, then $f'(x) = 1$ and $g(x) = e^x$, so

$$\int x \cdot e^x dx = x \cdot e^x - \int e^x dx$$
$$= x \cdot e^x - e^x$$

b. $\int x \cdot \ln(x)\, dx$

answer: Let $f(x) = \ln(x)$ and $g'(x) = x$, then $f'(x) = 1/x$ and $g(x) = x^2/2$, so

$$\int x \cdot \ln(x)\, dx = \frac{1}{2} x^2 \cdot \ln(x) - \int \frac{1}{2} x^2 \cdot \frac{1}{x} dx$$
$$= \frac{1}{2} x^2 \cdot \ln(x) - \int \frac{1}{2} x dx$$
$$= \frac{1}{2} x^2 \cdot \ln(x) - \frac{1}{4} x^2$$

section 20.4

1. Write down the approximation of the following integrals, with $h = 1/3$:

   a. $\int\limits_1^2 \ln(x)\,\mathrm{d}x$
   answer:
   $$\int\limits_1^2 \ln(x)\,\mathrm{d}x \approx \ln(1) + \frac{1}{3}\ln\left(1+\frac{1}{3}\right) + \frac{1}{3}\ln\left(1+\frac{2}{3}\right) + \frac{1}{3}\ln(2)$$

   b. $\int\limits_0^1 x^2 e^x \mathrm{d}x$
   answer:
   $$\int\limits_0^1 x^2 e^x \mathrm{d}x \approx 0 + \frac{1}{3}\cdot\frac{1}{9}e^{\frac{1}{3}} + \frac{1}{3}\cdot\frac{4}{9}e^{\frac{2}{3}} + \frac{1}{3}\cdot e$$

Chapter 21 Matrices and vectors

section 21.2

1. Carry out the following matrix multiplications:

   a.
$$\begin{pmatrix} 3 & 8 \\ -6 & 5 \end{pmatrix} \cdot \begin{pmatrix} 78 & 5 \\ 86 & 2 \end{pmatrix}$$

   answer:
$$\begin{pmatrix} 922 & 31 \\ -38 & -20 \end{pmatrix}$$

   b.
$$\begin{pmatrix} 2 & 3 \\ 6 & -5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

   answer:
$$\begin{pmatrix} 0 \\ 28 \end{pmatrix}$$

   c.
$$7 \cdot \begin{pmatrix} 4 & 5 & 1 \\ 8 & 9 & 12 \end{pmatrix}$$

   answer:
$$\begin{pmatrix} 28 & 35 & 7 \\ 56 & 63 & 84 \end{pmatrix}$$

   d.
$$\begin{pmatrix} -2 & -4 \end{pmatrix} \cdot \begin{pmatrix} 5 & 5 \\ 7 & 8 \end{pmatrix}$$

   answer:
$$\begin{pmatrix} -38 & -42 \end{pmatrix}$$

2.

section 21.3

1. Give the trace and the determinant of each of the following matrices:

   a.
$$\begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix}$$

   answer: trace: 2, determinant: -4

b.

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

answer: trace: -2, determinant: 0

c.

$$\begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix}$$

answer: trace: -2, determinant: -1

2. Give the trace and the determinant of each of the following matrices:

a.

$$\begin{pmatrix} -2a & b \\ 3 & 0 \end{pmatrix}$$

answer: trace: $-2a$, determinant: $-3b$

b.

$$\begin{pmatrix} a & 2 \\ 1 & a+1 \end{pmatrix}$$

answer: trace: $2a+1$, determinant: $a \cdot (a+1) - 2$

c.

$$\begin{pmatrix} a & -b \\ a & 3b \end{pmatrix}$$

answer: trace: $a+3b$, determinant: $3ab + ab = 4ab$

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